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On the charged meson pair theory of nuclear forces

Part I

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Introduction

The problem of giving a consistent theory for the nuclear forces has not yet — even in the simplest case — been finally solved. The main experimental facts to be accounted for are — in the simplest case of two nucleons — the binding energy and electric quadrupolemoment of the deuteron, the proton-neutron and the proton-proton scattering. The magnetic moments of the nucleons and the light nuclei are moreover very accurately determined and the problems of the saturation of the nuclear forces and β-radioactivity have to be kept in mind. Many of these data can phenomenologically be described by several different potentials; potential well, Gauss potential, meson potential, etc. These potentials must be spin-dependent and, at least to some extent, non-central. It may, however, be necessary to assume different ranges of the nuclear forces in the singlet and triplet state of the deuteron. To get a survey of the experimental facts, discussed from a theoretical point of view, we would refer the reader to Bethe, *Elementary Nuclear Theory* [47], and Rosenfeld, *Nuclear Forces* [48], where numerous references to the literature are given.

Several theories have been developed which describe the interaction between the nucleons by a field of limited range. This field is associated with a particle, a meson, the mass of which is between the mass of an electron and of a nucleon. Charged particles of intermediate mass have been observed in the cosmic rays and lately in rays produced in the Berkeley cyclotron. The simplest assumption is, thus, that the nuclear forces are associated with virtual transitions of single, charged mesons causing an interaction term which is linear in the meson field function. This gives, however, in the first approximation no force between similar particles whereas in fact at least in the ^1S-state the interaction between two nucleons is almost charge-independent. In the current field theories neutral mesons play, therefore, a very important rôle. Either the field is described by neutral mesons alone (neutral theory) or by a combination of neutral and charged mesons (symmetrical theory). As to the field theories, we would refer the reader to Wentzel, *Quantentheorie der Wellenfelder* [43], to surveys given by Pauli [41] and Wentzel [47] in the Reviews of Modern Physics, to Pauli, *Meson Theory of Nuclear Forces* [46], and to Rosenfeld's already quoted book.

Since direct experimental evidence in favour of the existence of the neutral meson is so far lacking, it may be of interest to try to develop a theory of the nuclear forces built only upon charged mesons. Such a possibility is given by the meson pair theories possibly combined with a charged linear theory. In these theories the nuclear forces are associated with virtual, simultaneous transitions of pairs of mesons, one with positive charge and one with negative. Thus we get an interaction term which is bilinear in the meson field function. Starting from FERMI's theory of the β-radioactivity analogous theories were developed previous to the meson theories by WENTZEL [37], GAMOW and TELLER [37] etc., using pairs of electrons. Then several meson pair theories have been presented. WENTZEL [41, 42], assuming scalar mesons (spin 0) and the simplest spin-independent interaction term between the meson field and the nucleons, calculates the interaction potential between the nucleons both rigorously and in an approximation of the second order perturbation. JAUCH and LOPES [44] have calculated the potential assuming a spin-dependent interaction, scalar mesons and perturbation theory. MARSHAK [40], MARSHAK and WEISSKOPF [41] have in the perturbation approximation made calculations assuming spin 1/2 mesons, while JAUCH [42], HOURIET [43], WIGNER, CRITCHFIELD and TELLER [39], CRITCHFIELD [41], CRITCHFIELD and LAMB [40], have given rigorous treatments. KLEIN [43], BRULIN and HJALMARS [45, 46] have treated the case of vector mesons (spin 1) with some different spin-dependent interactions in the perturbation approximation. Rigorous treatments of spin-independent interaction terms and calculations in the strong coupling approximation assuming spin-dependent interactions have been performed by PAULI and HU [45], assuming both scalar and vector mesons. In all these cases only charged mesons have been used. However, pair theories with both charged and neutral mesons have been presented by NOMA [46, 48] (spin 0 and spin 1/2 mesons) and BRULIN and HJALMARS [47] (spin 1 mesons). As a result of the general field theory developed by KLEIN [46] some interaction terms, bilinear in the meson field function, will occur.

The purpose of this paper is to give a more complete discussion of the charged scalar and vector meson pair theories of the nuclear forces. We shall, however, limit ourselves to a treatment of the problem in the perturbation approximation. In a paper, published at the same time as this, HJALMARS [49] deduces the expression for the contribution to the HAMILTONIAN, corresponding to the possible interaction terms between a pair of scalar or vector mesons and a nucleon, fulfilling the requirements of LORENTZ-invariance and charge-independence. In Chapter I of this paper the corresponding nuclear interaction potential is calculated in the perturbation of the second order, neglecting the recoil and assuming the nucleons to be fixed during the transmission of the field. Even the terms depending on the velocity of the nucleons are calculated in order to get an estimate of the errors which arise when these terms are neglected. The calculation is analogous to that used by KLEIN [43]. In Chapter II the question of convergence is treated in some particular cases, assuming that the nucleons have a finite extension. Further some criterions for the validity of the perturbation calculation are deduced. In Chapter III, which will be published later on in this journal, a quantitative analysis of the proton-proton scattering is made assuming a particular choice of pair interaction and of meson mass. The analysis would, however, be about the same with a different choice of mass and interaction term. The problem of the deuteron and of the neutron-proton scattering is only qualitatively discussed. As distinguished from the former case, forces arising from the linear charged meson theories, may play an important rôle here in addition to the pair forces, and thus, these cases are not so decisive for the possibilities of the pair theories.

Finally I wish to express my sincere thanks to my teacher Prof. O. KLEIN not only for having proposed these problems to me but also for his helpful guidance and inspiring instruction. Further I wish to thank Fil. lic. S. HJALMARS for close cooperation on performing this work, Fil. kand Mrs. B. MOBACK, who has carried out most of the numerical work, and Mrs. W. CAMERON for looking through the English translation. To the SWEDISH ATOMIC COMMITTEE, from which I have received grants during the course of this investigation, I should like to express my gratitude.

Chapter I

Derivation of the nuclear forces in the perturbation of the second order

§ 1. The HAMILTONIAN

Letting an index s indicate scalar mesons and an index v vector mesons, we use a scalar $U_s(\boldsymbol{r})$ and the three space components $U_{v_1}(\boldsymbol{r})$, $U_{v_2}(\boldsymbol{r})$, $U_{v_3}(\boldsymbol{r})$[1] of a four dimensional vector as the field functions of the scalar and vector meson fields respectively and put $P_s(\boldsymbol{r})$ and $\boldsymbol{P}_v(\boldsymbol{r})$ as their canonical conjugates satisfying the ordinary commutation rules:

$$[U_s(\boldsymbol{r}), P_s(\boldsymbol{r}')] = [U_s^*(\boldsymbol{r}), P_s^*(\boldsymbol{r}')] = i\hbar\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1}$$

or

$$[U_{v\nu}(\boldsymbol{r}),\ P_{v\mu}(\boldsymbol{r}')] = [U_{v\nu}^*(\boldsymbol{r}), P_{v\mu}^*(\boldsymbol{r}')] = i\hbar\,\delta_{\nu\mu}\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{2}$$

all other commutation relations being zero. The fourth component of $U_{v\mu}(\boldsymbol{r})$ can be expressed by the $P_{v\mu}(\boldsymbol{r})$:s in the approximation we shall use.

The wave functions of the nucleons $\psi_\nu(r)$ ($\nu = N$ indicates neutrons, $\nu = P$ protons) fulfill the JORDAN-WIGNER conditions of antisymmetric quantisation. We put (summing for $\nu = N$ and P)

$$\begin{aligned}
S_0(\boldsymbol{r}) &= \sum_\nu \psi_\nu^*(\boldsymbol{r})\,\psi_\nu(\boldsymbol{r}) \\
\boldsymbol{S}_0(\boldsymbol{r}) &= \sum_\nu \psi_\nu^*(\boldsymbol{r})\,\sigma\,\psi_\nu(\boldsymbol{r}) \\
S_i(\boldsymbol{r}) &= -\sum_\nu \psi_\nu^*(\boldsymbol{r})\,\varrho_i\,\psi_\nu(\boldsymbol{r}) \qquad (i = 1,\,2,\,3) \\
\boldsymbol{S}_i(\boldsymbol{r}) &= -\sum_\nu \psi_\nu^*(\boldsymbol{r})\,\varrho_i\,\sigma\,\psi_\nu(\boldsymbol{r}) \qquad (i = 1,\,2,\,3)
\end{aligned} \tag{3}$$

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[1] Or as a vector $\boldsymbol{U}(\boldsymbol{r})$. Space vectors are denoted by fat types.

where ϱ_1, ϱ_2, ϱ_3, and σ (components σ_1, σ_2, σ_3) are the ordinary DIRAC matrices.

We put further the meson mass $= \mu$, $\mu c/\hbar = \varkappa$; the nucleon mass $= M$, neglecting the mass difference between the proton and the neutron.

We write the HAMILTONIAN of the system of meson field and nucleons

$$H_s = H_{0s} + H_n + H_{is} \tag{4 s}$$

and

$$H_v = H_{0v} + H_n + H_{iv} \tag{4 v}$$

where the index 0 refers to the unperturbed meson field, the index n to the nucleons and the index i to the interaction between the meson field and the nucleons.

We have

$$H_n = -\sum_\nu \int \psi_\nu^*(\boldsymbol{r})\,(\varrho_1\,\sigma\cdot\boldsymbol{p}\,c + \varrho_3\,M\,c^2)\,\psi_\nu(\boldsymbol{r})\,d\,V. \tag{5}$$

As, however, the mass of the nucleon is large compared with that of the meson, we shall assume that we may neglect the motion of the nucleons (cf. e. g. KLEIN [43]). Thus we shall disregard the time-dependence of the wave-functions of the nucleons.

Further we have (see e. g. HJALMARS [49])

$$H_{0s} = \int\left\{8\,\pi\,c^2\,P_s(\boldsymbol{r})\,P_s^*(\boldsymbol{r}) + \frac{1}{8\,\pi}(\varkappa^2\,U_s^*(\boldsymbol{r})\,U_s(\boldsymbol{r}) + \right.$$
$$\left. + \operatorname{grad} U_s^*(\boldsymbol{r})\operatorname{grad} U_s(\boldsymbol{r}))\right\}d\,V \tag{6}$$

and

$$H_{0v} = \int\left\{\frac{8\,\pi\,c^2}{\varkappa^2}(\varkappa^2\,\boldsymbol{P}_v(\boldsymbol{r})\cdot\boldsymbol{P}_v^*(\boldsymbol{r}) + \operatorname{div}\boldsymbol{P}_v(\boldsymbol{r})\operatorname{div}\boldsymbol{P}_v^*(\boldsymbol{r})) + \right.$$
$$\left. + \frac{1}{8\,\pi}(\varkappa^2\,\boldsymbol{U}_v^*(\boldsymbol{r})\cdot\boldsymbol{U}_v(\boldsymbol{r}) + \operatorname{rot}\boldsymbol{U}_v^*(\boldsymbol{r})\cdot\operatorname{rot}\boldsymbol{U}_v(\boldsymbol{r}))\right\}d\,V. \tag{7}$$

We expand the U:s and P:s in a FOURIER-series, whose functions are periodic in a cube of volume V, where V tends to infinity.

In the scalar case we put

$$\begin{aligned} U_s(\boldsymbol{r}) &= -\frac{i}{\sqrt{V}}\sum_k \mu(\boldsymbol{r})\,\{\xi^*(\boldsymbol{k}) - \eta(\boldsymbol{k})\}\,e^{i\boldsymbol{k}\boldsymbol{r}} \\ U_s^*(\boldsymbol{r}) &= -\frac{i}{\sqrt{V}}\sum_k \mu(\boldsymbol{k})\,\{\eta^*(\boldsymbol{k}) - \xi(\boldsymbol{k})\}\,e^{-i\boldsymbol{k}\boldsymbol{r}} \end{aligned} \tag{8}$$

$$\begin{aligned} P_s(\boldsymbol{r}) &= \frac{1}{\sqrt{V}}\sum_k \lambda(\boldsymbol{k})\,\{\eta^*(\boldsymbol{k}) + \xi(\boldsymbol{k})\}\,e^{-i\boldsymbol{k}\boldsymbol{r}} \\ P_s^*(\boldsymbol{r}) &= \frac{1}{\sqrt{V}}\sum_k \lambda(\boldsymbol{k})\,\{\xi^*(\boldsymbol{k}) + \eta(\boldsymbol{k})\}\,e^{i\boldsymbol{k}\boldsymbol{r}} \end{aligned} \tag{9}$$

and

$$\operatorname{grad} U_s(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{k} \boldsymbol{k}\, \mu(\boldsymbol{k}) \{\xi^*(\boldsymbol{k}) - \eta(\boldsymbol{k})\}\, e^{i\boldsymbol{k}\boldsymbol{r}} \tag{10}$$

where

$$\mu(\boldsymbol{k}) = \sqrt{4\pi c \hbar}\, (k^2 + \varkappa^2)^{-\frac{1}{4}} \tag{11}$$

$$\lambda(\boldsymbol{k}) = \sqrt{\frac{\hbar}{16\pi c}}\, (k^2 + \varkappa^2)^{\frac{1}{4}} \tag{12}$$

and the following commutation rules are valid

$$[\xi(\boldsymbol{k}), \xi^*(\boldsymbol{k}')] = [\eta(\boldsymbol{k}), \eta^*(\boldsymbol{k}')] = \delta_{\boldsymbol{k}\boldsymbol{k}'} \tag{13}$$

all other commutation relations being zero. Introducing these expressions into (6) we get

$$H_{0s} = c\hbar \sum \sqrt{k^2 + \varkappa^2}\, (N^+(\boldsymbol{k}) + N^-(\boldsymbol{k}) + 1) \tag{14}$$

where

$$N^+(\boldsymbol{k}) = \xi^*(\boldsymbol{k})\, \xi(\boldsymbol{k})\,; \tag{15}$$

$$N^-(\boldsymbol{k}) = \eta^*(\boldsymbol{k})\, \eta(\boldsymbol{k})\,; \tag{16}$$

$N^+(\boldsymbol{k})$ and $N^-(\boldsymbol{k})$ are diagonal matrices with

$$(N^+ \,|\, N^+(\boldsymbol{k}) \,|\, N^+) = N^+ \geq 0 \tag{17}$$

$$(N^- \,|\, N^-(\boldsymbol{k}) \,|\, N^-) = N^- \geq 0 \tag{18}$$

where N^+ and N^- are integers. The only terms of $\xi(\boldsymbol{k})$ and $\xi^*(\boldsymbol{k})$ or of $\eta(\boldsymbol{k})$ and $\eta^*(\boldsymbol{k})$ different from zero are

$$(N^+ \,|\, \xi(\boldsymbol{k}) \,|\, N^+ + 1) = \sqrt{N^+ + 1} \qquad (N^+ + 1 \,|\, \xi^*(\boldsymbol{k}) \,|\, N^+) = \sqrt{N^+ + 1} \tag{19}$$

and

$$(N^- \,|\, \eta(\boldsymbol{k}) \,|\, N^- + 1) = \sqrt{N^- + 1} \qquad (N^- + 1 \,|\, \eta^*(\boldsymbol{k}) \,|\, N^-) = \sqrt{N^- + 1} \tag{20}$$

where we have disregarded the time dependence. Our unperturbed system can thus be described by $N^+(\boldsymbol{k})$, $N^-(\boldsymbol{k})$ and some quantum numbers, representing the nuclear system, which we do not need to specify any further. As we are interested in the lowest quantum state we shall put both

$$N^+ \quad \text{and} \quad N^- = 0. \tag{21}$$

In the vector case we put

$$\begin{aligned} \boldsymbol{U}_v(\boldsymbol{r}) &= \frac{-i}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \mu_\nu(\boldsymbol{k})\, \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\xi_\nu^*(\boldsymbol{k}) - \eta_\nu(\boldsymbol{k}))\, e^{i\boldsymbol{k}\boldsymbol{r}} \\ \boldsymbol{U}_v^*(\boldsymbol{r}) &= \frac{-i}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \mu_\nu(\boldsymbol{k})\, \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\eta_\nu^*(\boldsymbol{k}) - \xi_\nu(\boldsymbol{k}))\, e^{-i\boldsymbol{k}\boldsymbol{r}} \end{aligned} \tag{22}$$

$$\begin{aligned} \boldsymbol{P}_\nu(\boldsymbol{r}) &= \frac{1}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \lambda_\nu(\boldsymbol{k})\, \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\eta_\nu^*(\boldsymbol{k}) + \xi_\nu(\boldsymbol{k}))\, e^{-i\boldsymbol{k}\boldsymbol{r}} \\ \boldsymbol{P}_\nu^*(\boldsymbol{r}) &= \frac{1}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \lambda_\nu(\boldsymbol{k})\, \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\xi_\nu^*(\boldsymbol{k}) + \eta_\nu(\boldsymbol{k}))\, e^{i\boldsymbol{k}\boldsymbol{r}} \end{aligned} \tag{23}$$

$$\operatorname{div} \boldsymbol{U}_\nu(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \mu_\nu(\boldsymbol{k})\, \boldsymbol{k} \cdot \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\xi_\nu^*(\boldsymbol{k}) - \eta_\nu(\boldsymbol{k}))\, e^{i\boldsymbol{k}\boldsymbol{r}} \tag{24}$$

$$\operatorname{rot} \boldsymbol{U}_\nu(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \mu_\nu(\boldsymbol{k})\, \boldsymbol{k} \times \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\xi_\nu^*(\boldsymbol{k}) - \eta_\nu(\boldsymbol{k}))\, e^{i\boldsymbol{k}\boldsymbol{r}} \tag{25}$$

$$\operatorname{div} \boldsymbol{P}_\nu(\boldsymbol{r}) = \frac{-i}{\sqrt{V}} \sum_{k} \sum_{\nu=1}^{3} \lambda_\nu(\boldsymbol{k})\, \boldsymbol{k} \cdot \boldsymbol{\varepsilon}_\nu(\boldsymbol{k})\, (\eta_\nu^*(\boldsymbol{k}) + \xi_\nu(\boldsymbol{k}))\, e^{-i\boldsymbol{k}\boldsymbol{r}} \tag{26}$$

where[1] $\boldsymbol{\varepsilon}_\nu(\boldsymbol{k})$ are three mutually perpendicular unit vectors of which we chose $\boldsymbol{\varepsilon}_3(\boldsymbol{k})$ parallel to $\boldsymbol{k}$ i. e.

$$\boldsymbol{k} = k\, \boldsymbol{\varepsilon}_3(\boldsymbol{k}) \tag{27}$$

an

$$\begin{aligned} \mu_1(\boldsymbol{k}) = \mu_2(\boldsymbol{k}) &= \sqrt{4\pi\hbar c}\, (k^2 + \varkappa^2)^{-\frac{1}{4}} \\ \mu_3(\boldsymbol{k}) &= \sqrt{4\pi\hbar c}\, \frac{(k^2 + \varkappa^2)^{\frac{1}{4}}}{\varkappa} \end{aligned} \tag{28}$$

and

$$\begin{aligned} \lambda_1(\boldsymbol{k}) = \lambda_2(\boldsymbol{k}) &= \sqrt{\frac{\hbar}{16\pi c}}\, (k^2 + \varkappa^2)^{\frac{1}{4}} \\ \lambda_3(\boldsymbol{k}) &= \sqrt{\frac{\hbar}{16\pi c}} \cdot \varkappa\, (k^2 + \varkappa^2)^{-\frac{1}{4}}. \end{aligned} \tag{29}$$

The following commutation rules are valid

$$[\xi_\nu(\boldsymbol{k}), \xi_{\nu'}^*(\boldsymbol{k}')] = [\eta_\nu(\boldsymbol{k}), \eta_{\nu'}^*(\boldsymbol{k}')] = \delta_{\nu\nu'}\, \delta_{kk'} \tag{30}$$

all other commutation relations being equal to zero.

Thus we get

$$H_{0\nu} = c\hbar \sum_{k} \sum_{\nu=1}^{3} \sqrt{k^2 + \varkappa^2}\, (N_\nu^+(\boldsymbol{k}) + N_\nu^-(\boldsymbol{k}) + 1) \tag{31}$$

where

$$\begin{aligned} N_\nu^+(\boldsymbol{k}) &= \xi_\nu^*(\boldsymbol{k})\, \xi_\nu(\boldsymbol{k}) \\ N_\nu^-(\boldsymbol{k}) &= \eta_\nu^*(\boldsymbol{k})\, \eta_\nu(\boldsymbol{k}). \end{aligned} \tag{32}$$

The matrices $N_\nu^+(\boldsymbol{k})$, $N_\nu^-(\boldsymbol{k})$, $\xi_\nu(\boldsymbol{k})$, $\xi_\nu^*(\boldsymbol{k})$, $\eta_\nu(\boldsymbol{k})$, $\eta_\nu^*(\boldsymbol{k})$ have the same properties as the corresponding matrices in the scalar case (17)—(20) and we get the corresponding quantum numbers. As before we shall only consider the case $N_\nu^+ = N_\nu^- = 0$.

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[1] A point denotes a scalar product and a cross a vector product.

If we consider all interaction terms between a pair of scalar mesons and a nucleon, satisfying the requirements of LORENTZ-invariance and of charge-independence we get (cf. HJALMARS [49])

$$H_{is} = \frac{e^2}{\mu c^2} \int dV \cdot \tag{33}$$

$$\cdot \Big\{ C'_{s1} S_3(\boldsymbol{r})\, U_s^*(\boldsymbol{r})\, U_s(\boldsymbol{r}) + \tag{$S_{1'}$}$$

$$+ C''_{s1} S_3(\boldsymbol{r}) \left(\frac{1}{\varkappa^2} \operatorname{grad} U^*(\boldsymbol{r}) \operatorname{grad} U(\boldsymbol{r}) - \left(\frac{8 \pi c}{\varkappa} \right)^2 P(\boldsymbol{r}) P^*(\boldsymbol{r}) \right) + \tag{$S_{1''}$}$$

$$\begin{aligned} + i C'_{s2} \Big[& \frac{8 \pi c}{\varkappa} S_0(\boldsymbol{r}) (P(\boldsymbol{r}) U(\boldsymbol{r}) - U^*(\boldsymbol{r}) P^*(\boldsymbol{r})) + \\ & + \boldsymbol{S}_1(\boldsymbol{r}) \frac{1}{\varkappa} (\operatorname{grad} U^*(\boldsymbol{r}) \cdot U(\boldsymbol{r}) - U^*(\boldsymbol{r}) \operatorname{grad} U(\boldsymbol{r})) \Big] + \end{aligned} \tag{$S_{2'}$}$$

$$\begin{aligned} + C''_{s2} \Big[& \frac{8 \pi c}{\varkappa} S_0(\boldsymbol{r}) (P(\boldsymbol{r}) U(\boldsymbol{r}) + U^*(\boldsymbol{r}) P^*(\boldsymbol{r})) + \\ & + \boldsymbol{S}_1(\boldsymbol{r}) \frac{1}{\varkappa} (\operatorname{grad} U^*(\boldsymbol{r}) \cdot U(\boldsymbol{r}) + U^*(\boldsymbol{r}) \operatorname{grad} U(\boldsymbol{r})) \Big] + \end{aligned} \tag{$S_{2''}$}$$

$$\begin{aligned} + i C'_{s3} \Big[& \boldsymbol{S}_3(\boldsymbol{r}) \frac{1}{\varkappa^2} \operatorname{grad} U^*(\boldsymbol{r}) \times \operatorname{grad} U(\boldsymbol{r}) + \\ & + \frac{1}{\varkappa} \cdot \frac{8 \pi c}{\varkappa} \boldsymbol{S}_2(\boldsymbol{r}) (P(\boldsymbol{r}) \operatorname{grad} U(\boldsymbol{r}) - \operatorname{grad} U^*(\boldsymbol{r}) P^*(\boldsymbol{r})) \Big] \Big\} \end{aligned} \tag{$S_{3'}$}$$

where the C:s are arbitrary real constants, and for the moment the index s is neglected in $U_s(\boldsymbol{r})$ and $P_s(\boldsymbol{r})$. This expression, however, is only an approximation, correct up to and including the first power of C. We put

$$(k^2 + \varkappa^2)^{-\frac{1}{4}} = \bar{\mu}(\boldsymbol{k}) \tag{34}$$

$$\frac{(k^2 + \varkappa^2)^{\frac{1}{4}}}{\varkappa} = \bar{\lambda}(\boldsymbol{k}) \tag{35}$$

$$\begin{aligned} \alpha(\boldsymbol{k}'', \boldsymbol{k}') &= \eta^*(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') + \xi(\boldsymbol{k}'') \eta(\boldsymbol{k}') - \eta^*(\boldsymbol{k}'') \eta(\boldsymbol{k}') - \xi(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') \\ \bar{\alpha}(\boldsymbol{k}'', \boldsymbol{k}') &= \eta^*(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') + \xi(\boldsymbol{k}'') \eta(\boldsymbol{k}') + \eta^*(\boldsymbol{k}'') \eta(\boldsymbol{k}') + \xi(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') \\ \beta(\boldsymbol{k}'', \boldsymbol{k}') &= \eta^*(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') - \xi(\boldsymbol{k}'') \eta(\boldsymbol{k}') - \eta^*(\boldsymbol{k}'') \eta(\boldsymbol{k}') + \xi(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') \\ \bar{\beta}(\boldsymbol{k}'', \boldsymbol{k}') &= \eta^*(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') - \xi(\boldsymbol{k}'') \eta(\boldsymbol{k}') + \eta^*(\boldsymbol{k}'') \eta(\boldsymbol{k}') - \xi(\boldsymbol{k}'') \xi^*(\boldsymbol{k}') \end{aligned} \tag{36}$$

Introducing (8)—(12), (34)—(36) into (33) we get

$$H_{is} = \frac{4 \pi}{V} \frac{e^2}{\mu c^2} c \hbar \sum_{k'' k'} \int d V' e^{-i(k''-k') r'} \cdot \tag{37}$$

$$\cdot \Big\{ - C'_{s1} S_3(\boldsymbol{r}') \bar{\mu}(\boldsymbol{k}'') \bar{\mu}(\boldsymbol{k}') \alpha(\boldsymbol{k}'', \boldsymbol{k}') - \tag{$S_{1'}$}$$

$$- C''_{s1}\, S_3(\boldsymbol{r}') \left[\frac{\boldsymbol{k}'' \cdot \boldsymbol{k}'}{\varkappa^2}\, \bar{\mu}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \alpha(\boldsymbol{k}'', \boldsymbol{k}') + \bar{\lambda}(\boldsymbol{k}'')\, \bar{\lambda}(\boldsymbol{k}')\, \bar{\alpha}(\boldsymbol{k}'', \boldsymbol{k}') \right] + \tag{$S_{1''}$}$$

$$\begin{aligned} + C_{s2} \Big[& S_0(\boldsymbol{r}') \left(\bar{\lambda}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \beta(\boldsymbol{k}'', \boldsymbol{k}') - \bar{\mu}(\boldsymbol{k}'')\, \bar{\lambda}(\boldsymbol{k}')\, \bar{\beta}(\boldsymbol{k}'', \boldsymbol{k}') \right) - \\ & - \boldsymbol{S}_1(\boldsymbol{r}')\, \frac{1}{\varkappa} (\boldsymbol{k}'' + \boldsymbol{k}')\, \bar{\mu}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \alpha(\boldsymbol{k}'', \boldsymbol{k}') \Big] + \end{aligned} \tag{$S_{2'}$}$$

$$\begin{aligned} + i\, C''_{s2} \Big[& - S_0(\boldsymbol{r}') \left(\bar{\lambda}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \beta(\boldsymbol{k}'', \boldsymbol{k}') + \bar{\mu}(\boldsymbol{k}'')\, \bar{\lambda}(\boldsymbol{k}')\, \bar{\beta}(\boldsymbol{k}'', \boldsymbol{k}') \right) + \\ & + \boldsymbol{S}_1(\boldsymbol{r}')\, \frac{1}{\varkappa} (\boldsymbol{k}'' - \boldsymbol{k}')\, \bar{\mu}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \alpha(\boldsymbol{k}'', \boldsymbol{k}') \Big] + \end{aligned} \tag{$S_{2''}$}$$

$$\begin{aligned} + i\, C'_{s3} \Big[& - \boldsymbol{S}_3(\boldsymbol{r}')\, \frac{1}{\varkappa^2} (\boldsymbol{k}'' \times \boldsymbol{k}')\, \bar{\mu}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \alpha(\boldsymbol{k}'', \boldsymbol{k}') + \\ & + \boldsymbol{S}_2(\boldsymbol{r}')\, \frac{1}{\varkappa} \left(\boldsymbol{k}'\, \bar{\lambda}(\boldsymbol{k}'')\, \bar{\mu}(\boldsymbol{k}')\, \beta(\boldsymbol{k}'', \boldsymbol{k}') + \boldsymbol{k}''\, \bar{\mu}(\boldsymbol{k}'')\, \bar{\lambda}(\boldsymbol{k}')\, \bar{\beta}(\boldsymbol{k}'', \boldsymbol{k}') \right) \Big] \Big\} . \end{aligned} \tag{$S_{3'}$}$$

Similarly we get in the vector case

$$H_{iv} = \frac{e^2}{\mu c^2} \int d V \cdot \tag{38}$$

$$\cdot \Big\{ C'_{v1}\, S_3(\boldsymbol{r}) \left(\boldsymbol{U}^*(\boldsymbol{r})\, \boldsymbol{U}(\boldsymbol{r}) - \frac{1}{\varkappa^2} \left(\frac{8 \pi c}{\varkappa} \right)^2 \operatorname{div} \boldsymbol{P}(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r}) \right) + \tag{$V_{1'}$}$$

$$+ C''_{v1}\, S_3(\boldsymbol{r}) \left(\left(\frac{8 \pi c}{\varkappa} \right)^2 \boldsymbol{P}(\boldsymbol{r})\, \boldsymbol{P}^*(\boldsymbol{r}) - \frac{1}{\varkappa^2} \operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) \right) + \tag{$V_{1''}$}$$

$$\begin{aligned} + i\, C'_{v2} \Big[& S_0(\boldsymbol{r})\, \frac{8 \pi c}{\varkappa} \left(\boldsymbol{P}(\boldsymbol{r})\, \boldsymbol{U}(\boldsymbol{r}) - \boldsymbol{U}^*(\boldsymbol{r})\, \boldsymbol{P}^*(\boldsymbol{r}) \right) - \\ & - \boldsymbol{S}_1(\boldsymbol{r})\, \frac{1}{\varkappa} \Big(\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{U}(\boldsymbol{r}) + \boldsymbol{U}^*(\boldsymbol{r}) \times \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) + \\ & + \left(\frac{8 \pi c}{\varkappa} \right)^2 \left(\boldsymbol{P}(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r}) - \operatorname{div} \boldsymbol{P}(\boldsymbol{r}) \cdot \boldsymbol{P}^*(\boldsymbol{r}) \right) \Big) \Big] + \end{aligned} \tag{$V_{2'}$}$$

$$\begin{aligned} + C''_{v2} \Big[& S_0(\boldsymbol{r})\, \frac{8 \pi c}{\varkappa} \left(\boldsymbol{P}(\boldsymbol{r})\, \boldsymbol{U}(\boldsymbol{r}) + \boldsymbol{U}^*(\boldsymbol{r})\, \boldsymbol{P}^*(\boldsymbol{r}) \right) - \\ & - \boldsymbol{S}_1(\boldsymbol{r})\, \frac{1}{\varkappa} \Big(\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{U}(\boldsymbol{r}) - \boldsymbol{U}^*(\boldsymbol{r}) \times \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) + \\ & + \left(\frac{8 \pi c}{\varkappa} \right)^2 \left(\boldsymbol{P}(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r}) + \operatorname{div} \boldsymbol{P}(\boldsymbol{r})\, \boldsymbol{P}^*(\boldsymbol{r}) \right) \Big) \Big] + \end{aligned} \tag{$V_{2''}$}$$

$$\begin{aligned} + i\, C'_{v3} \Big[& \boldsymbol{S}_3(\boldsymbol{r}) \left(\left(\frac{8 \pi c}{\varkappa} \right)^2 \boldsymbol{P}(\boldsymbol{r}) \times \boldsymbol{P}^*(\boldsymbol{r}) - \frac{1}{\varkappa^2} \operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \times \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) \right) - \\ & - \boldsymbol{S}_2(\boldsymbol{r})\, \frac{1}{\varkappa} \cdot \frac{8 \pi c}{\varkappa} \left(\boldsymbol{P}(\boldsymbol{r}) \times \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) + \operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{P}^*(\boldsymbol{r}) \right) \Big] + \end{aligned} \tag{$V_{3'}$}$$

$$+ i\, C''_{v3} \Big[\boldsymbol{S}_3(\boldsymbol{r}) \cdot \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{U}(\boldsymbol{r}) - \\ - \boldsymbol{S}_2(\boldsymbol{r}) \frac{1}{\varkappa} \frac{8\pi c}{\varkappa} (\operatorname{div} \boldsymbol{P}(\boldsymbol{r}) \cdot \boldsymbol{U}(\boldsymbol{r}) - \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r})) \Big] + \tag{$V_{3''}$}$$

$$+ C'_{v4} \Big[S_0(\boldsymbol{r}) \frac{8\pi c}{\varkappa} \Big(\boldsymbol{P}(\boldsymbol{r}) \times \boldsymbol{U}(\boldsymbol{r}) - \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{P}^*(\boldsymbol{r}) - \\ - \frac{1}{\varkappa^2} (\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r}) + \operatorname{div} \boldsymbol{P}(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r})) \Big) + \\ + S_1(\boldsymbol{r}) \frac{1}{\varkappa} (\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \cdot \boldsymbol{U}(\boldsymbol{r}) + \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r})) \Big] + \tag{$V_{4'}$}$$

$$+ i\, C''_{v4} \Big[S_0(\boldsymbol{r}) \frac{8\pi c}{\varkappa} \Big((\boldsymbol{P}(\boldsymbol{r}) \times \boldsymbol{U}(\boldsymbol{r}) + \boldsymbol{U}^*(\boldsymbol{r}) \times \boldsymbol{P}^*(\boldsymbol{r})) - \\ - \frac{1}{\varkappa^2} (\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{div} \boldsymbol{P}^*(\boldsymbol{r}) - \operatorname{div} \boldsymbol{P}(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r})) \Big) + \\ + S_1(\boldsymbol{r}) \frac{1}{\varkappa} (\operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \cdot \boldsymbol{U}(\boldsymbol{r}) - \boldsymbol{U}^*(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r})) \Big] + \tag{$V_{4''}$}$$

$$+ C'_{v5} S_2(\boldsymbol{r}) \frac{8\pi c}{\varkappa^2} (\boldsymbol{P}(\boldsymbol{r}) \operatorname{rot} \boldsymbol{U}(\boldsymbol{r}) + \operatorname{rot} \boldsymbol{U}^*(\boldsymbol{r}) \cdot \boldsymbol{P}^*(\boldsymbol{r})) \Big\} \tag{$V_{5'}$}$$

where the C:s are arbitrary real constants and the index v in $\boldsymbol{U}_v(\boldsymbol{r})$ and $\boldsymbol{P}_v(\boldsymbol{r})$ is neglected. As above, this expression is correct up to the first power of C.

We put

$$\bar{\mu}_1(\boldsymbol{k}) = \bar{\mu}_2(\boldsymbol{k}) = (k^2 + \varkappa^2)^{-\frac{1}{4}} = \bar{\lambda}_3(\boldsymbol{k}) \\ \bar{\mu}_3(\boldsymbol{k}) = \frac{(k^2 + \varkappa^2)^{\frac{1}{4}}}{\varkappa} = \bar{\lambda}_1(\boldsymbol{k}) = \bar{\lambda}_2(\boldsymbol{k}) \tag{39}$$

and

$$\alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \eta^*_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') + \xi_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') - \eta^*_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') - \xi_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') \\ \bar{\alpha}_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \eta^*_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') + \xi_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') + \eta^*_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') + \xi_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') \\ \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \eta^*_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') - \xi_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') - \eta^*_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') + \xi_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') \\ \bar{\beta}_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \eta^*_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}') - \xi_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') + \eta^*_{r''}(\boldsymbol{k}'')\, \eta_{r'}(\boldsymbol{k}') - \xi_{r''}(\boldsymbol{k}'')\, \xi^*_{r'}(\boldsymbol{k}'). \tag{40}$$

Introducing (22)—(29), (39), (40) into (38) we get

$$H_{iv} = \frac{4\pi}{V} \frac{e^2}{\mu c^2} c\hbar \sum_{\boldsymbol{k}'', \boldsymbol{k}'} \sum_{r'', r' = 1}^{3} \int d\, V' e^{-i(\boldsymbol{k}'' - \boldsymbol{k}')\boldsymbol{r}'} \cdot \tag{41}$$

$$\cdot \Big\{ C'_{v1} S_3(\boldsymbol{r}') \Big(- \bar{\mu}_{r''}(\boldsymbol{k}'')\, \bar{\mu}_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') - \\ - \frac{1}{\varkappa^2} \bar{\lambda}_{r''}(\boldsymbol{k}'')\, \bar{\lambda}_{r'}(\boldsymbol{k}')\, (\boldsymbol{k}'' \cdot \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\, (\boldsymbol{k}' \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar{\alpha}_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Big) + \tag{$V_{1'}$}$$

$$\begin{aligned}
&+ C''_{v1} S_3(\boldsymbol{r}') \Bigl(\bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar\alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \frac{1}{\varkappa^2} \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')) \cdot (\boldsymbol{k}' \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) +
\end{aligned} \tag{$V_{1''}$}$$

$$\begin{aligned}
&+ C'_{v2} \Bigl[S_0(\boldsymbol{r}') \Bigl(\bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') - \\
&- \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar\beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) + \\
&+ \boldsymbol{S}_1(\boldsymbol{r}') \frac{1}{\varkappa} \Bigl(\bar\mu_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, \{ (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')) \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') - \\
&\qquad\qquad - \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times (\boldsymbol{k}' \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) \}\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, \{ \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\, (\boldsymbol{k}' \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) + (\boldsymbol{k}'' \cdot \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\, \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') \}\, \bar\alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) \Bigr] +
\end{aligned} \tag{$V_{2'}$}$$

$$\begin{aligned}
&+ i\, C''_{v2} \Bigl[S_0(\boldsymbol{r}') \Bigl(- \bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') - \\
&- \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar\beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) + \\
&+ \boldsymbol{S}_1(\boldsymbol{r}') \frac{1}{\varkappa} \Bigl(- \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, \{ (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')) \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') + \\
&\qquad\qquad + \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times (\boldsymbol{k}' \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) \}\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') - \\
&- \bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, \{ \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\, (\boldsymbol{k}' \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) - (\boldsymbol{k}'' \cdot \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\, \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') \}\, \bar\alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr] +
\end{aligned} \tag{$V_{2''}$}$$

$$\begin{aligned}
&+ i\, C'_{v3} \Bigl[\boldsymbol{S}_3(\boldsymbol{r}') \Bigl(\bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar\alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, \frac{1}{\varkappa^2} \{ (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')) \times (\boldsymbol{k}' \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) \}\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) + \\
&+ \boldsymbol{S}_2(\boldsymbol{r}') \frac{1}{\varkappa} \Bigl(- \bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, \{ \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times (\boldsymbol{k}' \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) \}\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, \{ (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')) \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') \}\, \bar\beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) \Bigr] +
\end{aligned} \tag{$V_{3'}$}$$

$$\begin{aligned}
&+ i\, C''_{v3} \Bigl[\boldsymbol{S}_3(\boldsymbol{r}') \Bigl(- \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) + \\
&+ \boldsymbol{S}_2(\boldsymbol{r}') \frac{1}{\varkappa} \Bigl(\bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, \{ (\boldsymbol{k}'' \cdot \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\, \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') \}\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, \{ \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\, (\boldsymbol{k}' \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')) \}\, \bar\beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') \Bigr) \Bigr] +
\end{aligned} \tag{$V_{3''}$}$$

$$\begin{aligned}
&+ i\, C'_{v4} \Bigl[\boldsymbol{S}_0(\boldsymbol{r}') \Bigl(- \bar\lambda_{r''}(\boldsymbol{k}'')\, \bar\mu_{r'}(\boldsymbol{k}')\, (\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'') \times \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') + \\
&+ \frac{1}{\varkappa^2} \bar\mu_{r''}(\boldsymbol{k}'')\, \bar\lambda_{r'}(\boldsymbol{k}')\, (\boldsymbol{k}'' \times \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\, (\boldsymbol{k}' \cdot \boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\, \bar\beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') +
\end{aligned}$$

$$\begin{aligned}
&+ \bar{\mu}_{r''}(\boldsymbol{k}'')\,\bar{\lambda}_{r'}(\boldsymbol{k}')\,(\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\bar{\beta}_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}') + \\
&+ \frac{1}{\varkappa^2}\,\bar{\lambda}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}(\boldsymbol{k}')\,(\boldsymbol{k}''\cdot\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}''))\,(\boldsymbol{k}'\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\beta_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}')) + \\
&+ S_1(\boldsymbol{r}')\,\frac{1}{\varkappa}\Big(\bar{\mu}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}(\boldsymbol{k}')\,\{(\boldsymbol{k}''\times\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\cdot\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') - \\
&\qquad\qquad - \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\cdot(\boldsymbol{k}'\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\}\;\alpha_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}')\Big)\Big] +
\end{aligned} \tag{$V_{4'}$}$$

$$\begin{aligned}
&+ C''_{v4}\Big[S_0(\boldsymbol{r}')\Big(\bar{\lambda}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}(\boldsymbol{k}')\,(\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\beta_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}') - \\
&- \frac{1}{\varkappa^2}\,\bar{\mu}_{r''}(\boldsymbol{k}'')\,\bar{\lambda}_{r'}(\boldsymbol{k}')\,(\boldsymbol{k}''\times\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\,(\boldsymbol{k}'\cdot\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\bar{\beta}_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}') + \\
&+ \bar{\mu}_{r''}(\boldsymbol{k}'')\,\lambda_{r'}(\boldsymbol{k}')\,(\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\bar{\beta}_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}') + \\
&+ \frac{1}{\varkappa^2}\,\bar{\lambda}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}(\boldsymbol{k}')\,(\boldsymbol{k}''\cdot\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\,(\boldsymbol{k}'\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\beta_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}')\Big) + \\
&+ S_1(\boldsymbol{r}')\,\frac{1}{\varkappa}\Big(-\bar{\mu}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}\cdot(\boldsymbol{k}')\,\{(\boldsymbol{k}''\times\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\,\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}') + \\
&+ \boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\cdot(\boldsymbol{k}'\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\}\;\alpha_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}')\Big)\Big] +
\end{aligned} \tag{$V_{4''}$}$$

$$\begin{aligned}
&+ C'_{v5}\,S_2(\boldsymbol{r}')\,\frac{1}{\varkappa}\Big(\bar{\lambda}_{r''}(\boldsymbol{k}'')\,\bar{\mu}_{r'}(\boldsymbol{k}')\,\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}'')\cdot(\boldsymbol{k}'\times\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}'))\,\beta_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}') - \\
&- \bar{\mu}_{r''}(\boldsymbol{k}'')\,\bar{\lambda}_{r'}(\boldsymbol{k}')\,(\boldsymbol{k}''\times\boldsymbol{\varepsilon}_{r''}(\boldsymbol{k}''))\cdot\boldsymbol{\varepsilon}_{r'}(\boldsymbol{k}')\,\beta_{r''r'}(\boldsymbol{k}'',\boldsymbol{k}')\Big)\Big\}.
\end{aligned} \tag{$V_{5'}$}$$

§ 2. The perturbation calculation

A. *Vector mesons*

We will begin with the vector meson interaction as the arguments, applied to this case, can easily be simplified so as to be valid in the scalar case too. As the perturbation of the first order does not give any interaction energy[1] we may limit ourselves to the perturbation of the second order. The diagonal elements of the matrix of the energy of the perturbation of the second order are

$$(n'\,|\,\Delta H\,|\,n') = \sum_{n''}\frac{(n'\,|\,H_i\,|\,n'')\,(n''\,|\,H_i\,|\,n')}{E_{n'}-E_{n''}} \tag{42}$$

where the quantum numbers n are given immediately after (32) (cf. (21)). The matrix H_i is given in (41) (or (37)). As in the quantum numbers n' in (41) (or (37)) $N^{+'}$ and $N^{-'}$ are put $=0$ and thus $N^{+''}$ and $N^{-''}$ are $=1$ (see (19) and (20)) the only terms in the matrix product (42) which are different from zero contain a factor (cf. (32) or (15) and (16))

$$\xi_{r''}(\boldsymbol{k}'')\,\eta_{r'}(\boldsymbol{k}')\,\eta^*_{s''}(\boldsymbol{l}'')\,\xi^*_{s'}(\boldsymbol{l}') = \delta_{r's''}\,\delta_{r''s'}\,\delta_{k'l''}\,\delta_{k''l'}. \tag{44}$$

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[1] The neglected C^2 terms in (38) (or (33)) give in some cases contact interaction terms. Such terms may be neglected, however, owing to the break down of the theory at small distances (cf. Chapters II and III).

From (31) (or (14)) we have

$$E_{n'} - E_{n''} = -\hbar c \left(\sqrt{k''^2 + \varkappa^2} + \sqrt{k'^2 + \varkappa^2}\right). \tag{45}$$

From (44) and (40) (resp. (36)) it follows that in (42) we may put $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ and we get

$$\alpha_{s''s'}(\boldsymbol{l}'', \boldsymbol{l}')\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \delta_{s''r'}\, \delta_{s'r''}\, \delta_{\boldsymbol{l}''\boldsymbol{k}'}\, \delta_{\boldsymbol{l}'\boldsymbol{k}''} \tag{46 a}$$

$$\alpha_{s''s'}(\boldsymbol{l}'', \boldsymbol{l}')\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = \delta_{s''r'}\, \delta_{s'r''}\, \delta_{\boldsymbol{l}''\boldsymbol{k}'}\, \delta_{\boldsymbol{l}'\boldsymbol{k}''} \tag{46 b}$$

$$\beta_{s''s'}(\boldsymbol{l}'', \boldsymbol{l}')\, \alpha_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = -\, \delta_{s''r'}\, \delta_{s'r''}\, \delta_{\boldsymbol{l}''\boldsymbol{k}'}\, \delta_{\boldsymbol{l}'\boldsymbol{k}''} \tag{46 c}$$

$$\beta_{s''s'}(\boldsymbol{l}'', \boldsymbol{l}')\, \beta_{r''r'}(\boldsymbol{k}'', \boldsymbol{k}') = -\, \delta_{s''r'}\, \delta_{s'r''}\, \delta_{\boldsymbol{l}''\boldsymbol{k}'}\, \delta_{\boldsymbol{l}'\boldsymbol{k}''}. \tag{46 d}$$

The interaction energy ΔH is obtained if in H_i ((41) or (37)) instead of the indices $\boldsymbol{k}''$, $\boldsymbol{k}'$, r'', r' we write $\boldsymbol{l}''$, $\boldsymbol{l}'$, s'', s' and multiply this expression with the original H_i where, however, instead of the variable $\boldsymbol{r}'$ we write $\boldsymbol{r}''$. According to (46) $\boldsymbol{l}''$, $\boldsymbol{l}'$, s'', s' are changed into $\boldsymbol{k}'$, $\boldsymbol{k}''$, s', s'', and in the case of (46 c) and (d) even the sign is changed. In practice this means that to get ΔH we have to multiply H_i (41 or 37) by itself, having in the first factor exchanged $\boldsymbol{k}''$ and $\boldsymbol{k}'$ and r'' and r', in the second factor changed the variable $\boldsymbol{r}'$ into $\boldsymbol{r}''$, and replaced the products of α and β with a plus or minus sign in accordance with (46 a, b) or (46 c, d). As in the sums and integrals the notation of the indices and variables is immaterial, we see that, if we change the notation suitably, many terms in ΔH occur twice. These terms have different signs if one term has $\alpha\beta$ as a factor while the other has $\beta\alpha$ (46 b and c). Thus a great many pairs of terms cancel each other and other terms double themselves. It is, particularly in the cross terms between different interactions, often useful to exchange the indices in some terms and alter the signs of $\boldsymbol{k}'$ and $\boldsymbol{k}''$, $\varepsilon_{r'}(\boldsymbol{k}')$ and $\varepsilon_{r''}(\boldsymbol{k}'')$. This means that the sign of the term is altered, if it has an odd number of factors $\boldsymbol{k}$. In this way, we shall again have some terms which either cancel one another or double themselves.

We shall now let the volume V tend to infinity replacing the summations with respect to $\boldsymbol{k}'$ and $\boldsymbol{k}''$ by integrals. In accordance with an earlier paper by Hjalmars and the present author [44] we shall multiply the integrand by a function $f(|\boldsymbol{k}'|, |\boldsymbol{k}''|, a)$ so that the integrals converge. a is a parameter (cf. also Noma [48]). In order to evaluate the $\boldsymbol{k}$-integrals we will introduce polar coordinates taking $\boldsymbol{r}'' - \boldsymbol{r}'$ as the polar axis. We shall use a rectangular coordinate system too, the z-axis being the same as the polar axis. The components of $\boldsymbol{k}$ in the polar coordinate system being (k, ϑ, φ) we may put, using (27)

$$\begin{aligned} \varepsilon_1(\boldsymbol{k}) &= \hat{x} \sin\varphi - \hat{y} \cos\varphi \\ \varepsilon_2(\boldsymbol{k}) &= \hat{x} \cos\vartheta \cos\varphi + \hat{y} \cos\vartheta \sin\varphi - \hat{z} \sin\vartheta \\ \varepsilon_3(\boldsymbol{k}) &= \hat{x} \sin\vartheta \cos\varphi + \hat{y} \sin\vartheta \sin\varphi + \hat{z} \cos\vartheta = \frac{1}{|\boldsymbol{k}|} \cdot \boldsymbol{k} \end{aligned} \tag{48}$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the coordinate unit vectors of the rectangular system.

We shall put

$$|\boldsymbol{k}'| = \varkappa p, \; |\boldsymbol{k}''| = \varkappa q, \; \cos\vartheta' = u, \; \cos\vartheta'' = v, \; |\boldsymbol{r}'' - \boldsymbol{r}'| = r \tag{49}$$

and shall use rather often the following elementary integrals

$$\int_0^{2\pi} \cos\varphi \, d\varphi = \int_0^{2\pi} \sin\varphi \, d\varphi = \int_0^{2\pi} \cos\varphi \sin\varphi \, d\varphi = 0 \tag{50 a}$$

$$\int_0^{2\pi} \cos^2\varphi \, d\varphi = \int_0^{2\pi} \sin^2\varphi \, d\varphi = \pi \tag{50 b}$$

$$\int_0^{2\pi} d\varphi = 2\pi. \tag{50 c}$$

From the integrations with respect to φ' and φ'' and from the arguments on the preceding page it follows that all terms of the interaction energy of the second order perturbation arising from different terms in (41) (or 37) are zero with exception of those arising from V_ν and $V_{\nu\pm 1}$ or S_ν and $S_{\nu\pm 1}$.

We get from (41)

$$\Delta H_v(a) = -\frac{1}{2\pi^2}\left(\frac{e^2}{\hbar c}\right)^2 \mu c^2 \iint dV' \, dV'' \int_0^\infty \int_0^\infty dp \, dq \cdot$$

$$\cdot \frac{f(p,q,a)\, p^2 q^2}{\sqrt{p^2+1}\sqrt{q^2+1}\,(\sqrt{p^2+1}+\sqrt{q^2+1})} \int_{-1}^{+1}\int_{-1}^{+1} e^{ikr(pu-qv)} \, du \, dv \cdot \tag{51}$$

$$\cdot \{C'^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') (6 + 2(p^2+q^2) + 3p^2q^2 - p^2q^2(u^2+v^2) + 3p^2q^2u^2v^2 +$$

$$+ 4pquv\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{1'1'})$$

$$+ C''^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') (6 + 4(p^2+q^2) + 6p^2q^2 - 2p^2q^2(u^2+v^2) + 6p^2q^2u^2v^2 +$$

$$+ 8pquv\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{1''1''})$$

$$+ C'_{v1} C''_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') (-12pquv - 12\sqrt{p^2+1}\sqrt{q^2+1}) +$$

$$+ C'^2_{v2} [S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') (12 + 6(p^2+q^2) + 2p^2q^2 + 2p^2q^2(u^2+v^2) - 6p^2q^2u^2v^2 -$$

$$- 12\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{1'1''})$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') (5(p^2+q^2) - (p^2u^2+q^2v^2) + 4p^2q^2 + 4p^2q^2u^2v^2 +$$

$$+ 8pquv\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{2'2'})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') (-(p^2+q^2) + 3(p^2u^2+q^2v^2) + 2p^2q^2 - 2p^2q^2(u^2+v^2) -$$

$$- 6p^2q^2u^2v^2 - 8pquv\sqrt{p^2+1}\sqrt{q^2+1} +$$

$$+ 12pquv)] +$$

$$+ C_{v2}^{\prime\prime 2} [S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') (12 + 6(p^2+q^2) + 2p^2q^2 + 2p^2q^2(u^2+v^2) - 6p^2q^2u^2v^2 +$$

$$+ 12\sqrt{p^2+1}\sqrt{q^2+1}) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') (5(p^2+q^2) - (p^2u^2+q^2v^2) + 4p^2q^2 + 4p^2q^2u^2v^2 +$$

$$+ 8pquv\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{2''2''})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') (-(p^2+q^2) + 3(p^2u^2+q^2v^2) - 2p^2q^2(u^2+v^2) - 6p^2q^2u^2v^2 -$$

$$- 8pquv\sqrt{p^2+1}\sqrt{q^2+1} -$$

$$- 12pquv)] +$$

$$+ C_{v3}^{\prime 2} [\boldsymbol{S}_3(\boldsymbol{r}') \boldsymbol{S}_3(\boldsymbol{r}'') (4 + 3(p^2+q^2) - (p^2u^2+q^2v^2) + 4p^2q^2 - 4p^2q^2u^2v^2) +$$

$$+ S_{3z}(\boldsymbol{r}') S_{3z}(\boldsymbol{r}'') (-(p^2+q^2) + 3(p^2u^2+q^2v^2) - 2p^2q^2 + 2p^2q^2(u^2+v^2) +$$

$$+ 6p^2q^2u^2v^2 + 8pquv\sqrt{p^2+1}\sqrt{q^2+1}) + \qquad (V_{3'3'})$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}') \boldsymbol{S}_2(\boldsymbol{r}'') (3(p^2+q^2) - (p^2u^2+q^2v^2) + 4p^2q^2 - 4p^2q^2u^2v^2) +$$

$$+ S_{2z}(\boldsymbol{r}') S_{2z}(\boldsymbol{r}'') (-(p^2+q^2) + 3(p^2u^2+q^2v^2) - 2p^2q^2 + 2p^2q^2(u^2+v^2) +$$

$$+ 6p^2q^2u^2v^2 +$$

$$+ 8pquv\sqrt{p^2+1}\sqrt{q^2+1})] +$$

$$+ C_{v3}^{\prime\prime 2} [\boldsymbol{S}_3(\boldsymbol{r}') \boldsymbol{S}_3(\boldsymbol{r}'') (4 + (p^2+q^2) + (p^2u^2+q^2v^2) + p^2q^2(u^2+v^2) - 2p^2q^2u^2v^2) +$$

$$+ S_{3z}(\boldsymbol{r}') S_{3z}(\boldsymbol{r}'') ((p^2+q^2) - 3(p^2u^2+q^2v^2) + p^2q^2 - 2p^2q^2(u^2+v^2) + 3p^2q^2u^2v^2) +$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}') \boldsymbol{S}_2(\boldsymbol{r}'') (2(p^2+q^2) + 2p^2q^2 - p^2q^2(u^2+v^2)) + \qquad (V_{3''3''})$$

$$+ S_{2z}(\boldsymbol{r}') S_{2z}(\boldsymbol{r}'') (-2p^2q^2 + 3p^2q^2(u^2+v^2) + 4pquv\sqrt{p^2+1}\sqrt{q^2+1})] +$$

$$+ C_{v3}' C_{v3}'' [\boldsymbol{S}_3(\boldsymbol{r}') \boldsymbol{S}_3(\boldsymbol{r}'') (-8\sqrt{p^2+1}\sqrt{q^2+1}) +$$

$$+ S_{3z}(\boldsymbol{r}') S_{3z}(\boldsymbol{r}'') (-8pquv) + \qquad (V_{3'3''})$$

$$+ (\boldsymbol{S}_2(\boldsymbol{r}') \boldsymbol{S}_2(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}') S_{2z}(\boldsymbol{r}''))(8pquv)] +$$

$$+ C_{v4}^{\prime 2} [\boldsymbol{S}_0(\boldsymbol{r}') \boldsymbol{S}_0(\boldsymbol{r}'') (8 + 4(p^2+q^2) + 4p^2q^2 + 4p^2q^2u^2v^2 +$$

$$+ 8pquv\sqrt{p^2+1}\sqrt{q^2+1} -$$

$$- 8pquv - 8\sqrt{p^2+1}\sqrt{q^2+1}) +$$

$$+ S_{0z}(\boldsymbol{r}') S_{0z}(\boldsymbol{r}'') (2p^2q^2 - 2p^2q^2(u^2+v^2) - 6p^2q^2u^2v^2 -$$

$$- 8pquv\sqrt{p^2+1}\sqrt{q^2+1} + \qquad (V_{4'4'})$$

$$+ 8pquv) +$$

$$+ S_1(\boldsymbol{r}')\,S_1(\boldsymbol{r}'')\big(4\,(p^2+q^2) + 2\,p^2q^2 + 2\,p^2q^2\,(u^2+v^2) - 6\,p^2q^2u^2v^2 +$$

$$+ 8\,pquv)\big] +$$

$$+ C''^2_{v4}\big[\boldsymbol{S}_0(\boldsymbol{r}')\,\boldsymbol{S}_0(\boldsymbol{r}'')\big(8 + 4\,(p^2+q^2) + 4\,p^2q^2 + 4\,p^2q^2u^2v^2 +$$

$$+ 8\,pquv\sqrt{p^2+1}\sqrt{q^2+1} +$$

$$+ 8\,pquv + 8\sqrt{p^2+1}\sqrt{q^2+1}\big) + \qquad (V_{4''4''})$$

$$+ S_{0z}(\boldsymbol{r}')\,S_{0z}(\boldsymbol{r}'')\big(2\,p^2q^2 - 2\,p^2q^2\,(u^2+v^2) - 6\,p^2q^2u^2v^2 -$$

$$- 8\,pquv\sqrt{p^2+1}\sqrt{q^2+1} -$$

$$- 8\,pquv\big) +$$

$$+ S_1(\boldsymbol{r}')\,S_1(\boldsymbol{r}'')\big(4\,(p^2+q^2) + 2\,p^2q^2 + 2\,p^2q^2\,(u^2+v^2) - 6\,p^2q^2u^2v^2 -$$

$$- 8\,pquv)\big] +$$

$$+ C'^2_{v5}\,S_2(\boldsymbol{r}')\,S_2(\boldsymbol{r}'')\big(4\,(p^2+q^2) + 6\,p^2q^2 - 2\,p^2q^2\,(u^2+v^2) + 6\,p^2q^2u^2v^2 +$$

$$+ 8\,pquv\sqrt{p^2+1}\sqrt{q^2+1}\big) + \qquad (V_{5'5'})$$

$$+ i\,C'_{v1}\,C''_{v2}\big(S_3(\boldsymbol{r}')\,S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}')\,S_3(\boldsymbol{r}'')\big)\cdot\big(4\,(pu - qv) +$$

$$+ 2\,pquv\,(pu-qv) + 2\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big) + \qquad (V_{1'2''})$$

$$+ i\,C''_{v1}\,C''_{v2}\big(S_3(\boldsymbol{r}')\,S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}')\,S_3(\boldsymbol{r}'')\big)\cdot\big(-2\,(pu-qv) -$$

$$- 4\,pquv\,(pu-qv) - 4\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big) + \qquad (V_{1''2''})$$

$$+ i\,C'_{v2}\,C'_{v3}\big[\big(S_0(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}')\,S_0(\boldsymbol{r}'')\big)\cdot\big(-4\,(pu-qv) +$$

$$+ 2\,p^2qv - 2\,pq^2u + 2\,pquv\,(pu-qv) + 4\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big) +$$

$$+ \big(\boldsymbol{S}_1(\boldsymbol{r}')\times\boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}')\times\boldsymbol{S}_1(\boldsymbol{r}'')\big)_z\cdot\big(-2\,(pu-qv) - \qquad (V_{2'3'})$$

$$- p^2qv + pq^2u + 3\,pquv\,(pu-qv) + 2\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big)\big] +$$

$$+ i\,C'_{v2}\,C''_{v3}\big[\big(S_0(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}')\,S_0(\boldsymbol{r}'')\big)\cdot\big(-2\,(pu-qv) +$$

$$+ 2\,pquv\,(pu-qv) + 2\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big) +$$

$$+ \big(\boldsymbol{S}_1(\boldsymbol{r}')\times\boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}')\times\boldsymbol{S}_1(\boldsymbol{r}'')\big)_z\cdot\big(-2\,(pu-qv) + \qquad (V_{2'3''})$$

$$+ 2\,pquv\,(pu-qv) + 2\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big)\big] +$$

$$+ i\,C'_{v3}\,C''_{v4}\big[\big(S_{3z}(\boldsymbol{r}')\,S_1(\boldsymbol{r}'') - S_1(\boldsymbol{r}')\,S_{3z}(\boldsymbol{r}'')\big)\big(-2\,p^2qv + 2\,pq^2u -$$

$$- 2\,pquv\,(pu-qv) - 4\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big) + \qquad (V_{3'4''})$$

$$+ \big(\boldsymbol{S}_2(\boldsymbol{r}')\times\boldsymbol{S}_0(\boldsymbol{r}'') + \boldsymbol{S}_0(\boldsymbol{r}')\times\boldsymbol{S}_2(\boldsymbol{r}'')\big)_z\cdot\big(-2\,(pu-qv) +$$

$$+ p^2qv - pq^2u - 3\,pquv\,(pu-qv) - 2\,(pu-qv)\sqrt{p^2+1}\sqrt{q^2+1}\big)\big] +$$

$$+ i\, C''_{v3}\, C''_{v4} \left[\left(S_{3z}(\mathbf{r}')\, S_1(\mathbf{r}'') - S_1(\mathbf{r}')\, S_{3z}(\mathbf{r}'') \right) \cdot \left(+\, 4\,(p\,u - q\,v) - \right.\right.$$

$$\left. - 2\,p^2\,q\,v + 2\,p\,q^2\,u + 2\,p\,q\,u\,v\,(p\,u - q\,v) \right) + \qquad (V_{3''4''})$$

$$+ \left(\boldsymbol{S}_2(\mathbf{r}') \times \boldsymbol{S}_0(\mathbf{r}'') + \boldsymbol{S}_0(\mathbf{r}') \times \boldsymbol{S}_2(\mathbf{r}'') \right)_z \cdot \left(2\,(p\,u - q\,v) + \right.$$

$$\left.\left. + p^2\,q\,v - p\,q^2\,u + p\,q\,u\,v\,(p\,u - q\,v) + 2\,(p\,u - q\,v) \sqrt{p^2+1}\sqrt{q^2+1} \right) \right] +$$

$$+ i\, C'_{v4}\, C'_{v5} \left(S_{0z}(\mathbf{r}')\, S_2(\mathbf{r}'') - S_2(\mathbf{r}')\, S_{0z}(\mathbf{r}'') \right) \cdot \left(-\, 4\,(p\,u - q\,v) + \right.$$

$$\left. + 4\,p\,q\,u\,v\,(p\,u - q\,v) + 4\,(p\,u - q\,v) \sqrt{p^2+1}\sqrt{q^2+1} \right) \} \qquad (V_{4'5'})$$

all other interaction terms being zero.

In this equation we put $\varkappa r = x$ in $\varkappa r p u$ and $\varkappa r = y$ in $\varkappa r q v$. The function $f(p, q, \alpha)$ in (51) is assumed to tend to zero as p or q tend to infinity if $\alpha \neq 0$, so that the integrals in (51) converge uniformely with regard to x and y through the entire space.

Further

$$\lim_{\alpha = 0} f(p, q, \alpha) \equiv 1. \qquad (52)$$

We might e. g. put $f(p, q, \alpha) = e^{-\alpha(p+q)}$.

If we write the integrand of (51) as derivatives with regard to x and y it is permissible to remove the differential operators outside the integral signs as the integrals are uniformely convergent.

We shall use the following integrals

$$I_1(x, y, \alpha) =$$

$$= \tfrac{1}{4}\, x\, y \int_0^\infty \int_0^\infty \frac{p^2\, q^2\, d\,p\, d\,q\, f(p, q, \alpha)}{\sqrt{p^2+1}\sqrt{q^2+1}\left(\sqrt{p^2+1} + \sqrt{q^2+1}\right)} \int_{-1}^{+1} \int_{-1}^{+1} d\,u\, d\,v\, e^{i(xpu - yqv)} =$$

$$= \int_0^\infty \int_0^\infty \frac{p\, q\, d\,p\, d\,q\, f(p, q, \alpha)}{\sqrt{p^2+1}\sqrt{q^2+1}\left(\sqrt{p^2+1} + \sqrt{q^2+1}\right)} \sin x\,p \sin y\,q \qquad (53)$$

and

$$I_2(x, y, \alpha) =$$

$$= \tfrac{1}{4}\, x\, y \int_0^\infty \int_0^\infty \frac{p^2\, q^2\, d\,p\, d\,q\, f(p, q, \alpha)}{\sqrt{p^2+1} + \sqrt{q^2+1}} \int_{-1}^{+1} \int_{-1}^{+1} d\,u\, d\,v\, e^{i(xpu - yqv)} =$$

$$= \int_0^\infty \int_0^\infty \frac{p\, q\, d\,p\, d\,q\, f(p, q, \alpha)}{\sqrt{p^2+1} + \sqrt{q^2+1}} \sin x\,p \sin y\,q. \qquad (54)$$

Then we can write (51) as derivatives of $I_1(x, y, \alpha)$, $I_2(x, y, \alpha)$, $\dfrac{1}{xy} I_1(x, y, \alpha)$ and $\dfrac{1}{xy} I_2(x, y, \alpha)$ i. e.

$$\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty \frac{p^2 q^2\, d p\, d q\, f(p, q, \alpha)}{\sqrt{p^2+1}\sqrt{q^2+1}\,(\sqrt{p^2+1}+\sqrt{q^2+1})}\int\limits_{-1}^{+1}\int\limits_{-1}^{+1} d u\, d v\, e^{i(xpu-yqv)}\,.$$

$$\cdot\, p^2 u^2 = -\frac{\partial^2}{\partial x^2}\frac{1}{xy} I_1(x, y, \alpha) \qquad (55)$$

$$\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty \frac{p^2 q^2\, d p\, d q\, f(p, q, \alpha)}{\sqrt{p^2+1}\sqrt{q^2+1}\,(\sqrt{p^2+1}+\sqrt{q^2+1})}\, q^2 \int\limits_{-1}^{+1}\int\limits_{-1}^{+1} d u\, d v\, e^{i(xpu-yqv)} =$$

$$= \frac{1}{xy}\int\limits_0^\infty\int\limits_0^\infty \frac{p q\, d p\, d q\, f(p, q, \alpha)}{\sqrt{p^2+1}\sqrt{q^2+1}\,(\sqrt{p^2+1}+\sqrt{q^2+1})}\, q^2 \sin x p \sin y q =$$

$$= -\frac{1}{xy}\frac{\partial^2}{\partial y^2} I_1(x, y, \alpha) \qquad (56)$$

and correspondingly for the other terms of (51).

Thus we get $\Delta H_v(\alpha)$ expressed as derivatives of (53) and (54)

$$\Delta H_v(\alpha) = -\frac{2}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^2 \mu c^2 \int\int d V'\, d V''. \qquad (57)$$

$$\cdot \biggl\{ C'^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \biggl(\biggl\{\frac{6}{xy} - \frac{2}{xy}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{3}{xy}\frac{\partial^4}{\partial x^2\,\partial y^2} - \biggl(\frac{\partial^2}{\partial x^2}\frac{1}{xy}\frac{\partial^2}{\partial y^2} + \qquad (V_{1'1'})$$

$$+ \frac{\partial^2}{\partial y^2}\frac{1}{xy}\frac{\partial^2}{\partial x^2}\biggr) + 3\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\biggr\} I_1(x, y, \alpha) + 4\frac{\partial^2}{\partial x\,\partial y}\frac{1}{xy} I_2(x, y, \alpha) +$$

$$+ C''^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \biggl(\biggl\{\frac{6}{xy} - \frac{4}{xy}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{6}{xy}\frac{\partial^4}{\partial x^2\,\partial y^2} - 2\biggl(\frac{\partial^2}{\partial x^2}\frac{1}{xy}\frac{\partial^2}{\partial y^2} + \qquad (V_{1''1''})$$

$$+ \frac{\partial^2}{\partial y^2}\frac{1}{xy}\frac{\partial^2}{\partial x^2}\biggr) + 6\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\biggr\} I_1(x, y, \alpha) + 8\frac{\partial^2}{\partial x y}\frac{1}{xy} I_2(x, y, \alpha)\biggr) +$$

$$+ C'_{v1} C''_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \biggl(-12\frac{\partial^2}{\partial x\,\partial y}\frac{1}{xy} I_1(x, y, \alpha) - \frac{12}{xy} I_2(x, y, \alpha)\biggr) + \qquad (V_{1'1''})$$

$$+ C'^2_{v2} \biggl[S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') \biggl(\biggl\{\frac{12}{xy} - \frac{6}{xy}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{2}{xy}\frac{\partial^4}{\partial x^2\,\partial y^2} + 2\biggl(\frac{\partial^2}{\partial x^2}\frac{1}{xy}\frac{\partial^2}{\partial y^2} +$$

$$+ \frac{\partial^2}{\partial y^2}\frac{1}{xy}\frac{\partial^2}{\partial x^2}\biggr) - 6\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\biggr\} I_1(x, y, \alpha) -$$

$$- \frac{12}{xy} I_2(x, y, \alpha)\biggr) + \qquad (V_{2'2'})$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') \biggl(\biggl\{-\frac{5}{xy}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} + \frac{4}{xy}\frac{\partial^4}{\partial x^2\,\partial y^2} +$$

$$+ 4\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\biggr\} I_1(x, y, \alpha) + 8\frac{\partial^2}{\partial x\,\partial y}\frac{1}{xy} I_2(x, y, \alpha)\biggr) +$$

$$+ S_{1z}(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'') \Bigg(\Bigg\{\frac{1}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 3\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} + \frac{2}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} -$$

$$- 2\left(\frac{\partial^2}{\partial x^2}\frac{1}{xy}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\frac{1}{xy}\frac{\partial^2}{\partial x^2}\right) - 6\,\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\Bigg\} I_1(x, y, \alpha) - 8\,\frac{\partial^2}{\partial x\,\partial y}\frac{1}{xy} I_2(x, y, \alpha) +$$

$$+ 12\,\frac{\partial^2}{\partial x\,\partial y}\,\frac{1}{xy} I_1(x, y, \alpha)\Bigg)\Bigg] +$$

$$+ C_{\nu 2}^{\prime\prime 2}\Bigg[S_0(\boldsymbol{r}')\, S_0(\boldsymbol{r}'')\Bigg(\Bigg\{\frac{12}{xy} - \frac{6}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{2}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} + 2\left(\frac{\partial^2}{\partial x^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial y^2} +\right.$$

$$\left.+ \frac{\partial^2}{\partial y^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial x^2}\right) - 6\,\frac{\partial^4}{\partial x^2\,\partial y^2}\,\frac{1}{xy}\Bigg\} I_1(x, y, \alpha) +$$

$$+ \frac{12}{xy} I_2(x, y, \alpha)\Bigg) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}')\, \boldsymbol{S}_1(\boldsymbol{r}'')\Bigg(\Bigg\{- \frac{5}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} + \frac{4}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} +$$

$$+ 4\,\frac{\partial^4}{\partial x^2\,\partial y^2}\,\frac{1}{xy}\Bigg\} I_1(x, y, \alpha) + 8\,\frac{\partial^2}{\partial x\,\partial y}\,\frac{1}{xy} I_2(x, y, \alpha)\Bigg) + \qquad (V_{2''2''})$$

$$+ S_{1z}(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'')\Bigg(\Bigg\{\frac{1}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 3\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} - 2\left(\frac{\partial^2}{\partial x^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial y^2} +\right.$$

$$\left.+ \frac{\partial^2}{\partial y^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial x^2}\right) - 6\,\frac{\partial^4}{\partial x^2\,\partial y^2}\,\frac{1}{xy}\Bigg\} I_1(x, y, \alpha) - 8\,\frac{\partial^2}{\partial x\,\partial y}\,\frac{1}{xy} I_2(x, y, \alpha) -$$

$$- 12\,\frac{\partial^2}{\partial x\,\partial y}\,\frac{1}{xy} I_1(x, y, \alpha)\Bigg)\Bigg] +$$

$$+ C_{\nu 3}^{\prime 2}\Bigg[\boldsymbol{S}_3(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'')\Bigg(\frac{4}{xy} - \frac{3}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} +$$

$$+ \frac{4}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} - 4\,\frac{\partial^4}{\partial x^2\,\partial y^2}\,\frac{1}{xy}\Bigg) I_1(x, y, \alpha) +$$

$$+ S_{3z}(\boldsymbol{r}')\, S_{3z}(\boldsymbol{r}'')\Bigg(\Bigg\{\frac{1}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 3\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} - \frac{2}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} +$$

$$+ 2\left(\frac{\partial^2}{\partial x^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\,\frac{1}{xy}\,\frac{\partial^2}{\partial x^2}\right) + 6\,\frac{\partial^4}{\partial x^2\,\partial y^2}\frac{1}{xy}\Bigg\} I_1(x, y, \alpha) +$$

$$+ 8\,\frac{\partial^2}{\partial x\,\partial y}\,\frac{1}{xy} I_2(x, y, \alpha)\Bigg) +$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'')\Bigg(- \frac{3}{xy}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{xy} + \qquad (V_{3'3'})$$

$$+ \frac{4}{xy}\,\frac{\partial^4}{\partial x^2\,\partial y^2} - 4\,\frac{\partial^4}{\partial x^2\,\partial y^2}\,\frac{1}{xy}\Bigg) I_1(x, y, \alpha) +$$

$$+ S_{2z}(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'') \biggl(\biggl\{ \frac{1}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 3 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{1}{xy} - \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} +$$

$$+ 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \biggr\} I_1(x, y, \alpha) +$$

$$+ 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) \biggr) \biggr] +$$

$$+ C_{v3}''^2 \biggl[\boldsymbol{S}_3(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'') \biggl(\frac{4}{xy} - \frac{1}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{1}{xy} +$$

$$+ \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) - 2 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \biggr) I_1(x, y, \alpha) +$$

$$+ S_{3z}(\boldsymbol{r}')\, S_{3z}(\boldsymbol{r}'') \biggl(- \frac{1}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 3 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{1}{xy} +$$

$$+ \frac{1}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} - 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) + 3 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \biggr) I_1(x, y, \alpha) +$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'') \biggl(- \frac{2}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} - \qquad (V_{3''3''})$$

$$- \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) \biggr) I_1(x, y, \alpha) +$$

$$+ S_{2z}(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'') \biggl(\biggl\{ - \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} + 3 \biggl(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} +$$

$$+ \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \biggr) \biggr\} I_1(x, y, \alpha) + 4 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) \biggr) \biggr] +$$

$$+ C_{v3}' C_{v3}'' \biggl[\boldsymbol{S}_3(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'') \left(- \frac{8}{xy} I_2(x, y, \alpha) \right) +$$

$$+ S_{3z}(\boldsymbol{r}')\, S_{3z}(\boldsymbol{r}'') \left(- 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) \right) + \qquad (V_{3'3''})$$

$$+ (\boldsymbol{S}_2(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'')) \left(8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) \right) \biggr] +$$

$$+ C_{v4}'^2 \biggl[\boldsymbol{S}_0(\boldsymbol{r}')\, \boldsymbol{S}_0(\boldsymbol{r}'') \biggl(\biggl\{ \frac{8}{xy} - \frac{4}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{4}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} +$$

$$+ 4 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \biggr\} I_1(x, y, \alpha) + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) -$$

$$- 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) - \frac{8}{xy} I_2(x, y, \alpha) \biggr) + \qquad (V_{4'4'})$$

$$+ S_{0z}(\boldsymbol{r}')\, S_{0z}(\boldsymbol{r}'') \left(\left\{ \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} - 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) - \right.\right.$$

$$\left. - 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \right\} I_1(x, y, \alpha) - 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) +$$

$$\left. + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) \right) +$$

$$+ S_1(\boldsymbol{r}')\, S_1(\boldsymbol{r}'') \left(- \frac{4}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} + 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \right.\right.$$

$$\left. + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) - 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} +$$

$$\left.\left. + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} \right) I_1(x, y, \alpha) \right] +$$

$$+ C''_{v4} \left[\boldsymbol{S}_0(\boldsymbol{r}')\, \boldsymbol{S}_0(\boldsymbol{r}'') \left(\left\{ \frac{8}{xy} - \frac{4}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{4}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} + \right.\right.\right.$$

$$\left. + 4 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \right\} I_1(x, y, \alpha) + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) +$$

$$\left. + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) + \frac{8}{xy} I_2(x, y, \alpha) \right) +$$

$$+ S_{0z}(\boldsymbol{r}')\, S_{0z}(\boldsymbol{r}'') \left(\left\{ \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} - 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) - \right.\right.$$

$$\left. - 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \right\} I_1(x, y, \alpha) - 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) - \qquad (V_{4''4''})$$

$$\left. - 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_1(x, y, \alpha) \right) +$$

$$+ S_1(\boldsymbol{r}')\, S_1(\boldsymbol{r}'') \left(- \frac{4}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{2}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} + 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \right.\right.$$

$$\left. + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) - 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} -$$

$$\left.\left. - 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} \right) I_1(x, y, \alpha) \right] +$$

$$+ C'^2_{v5}\, S_2(\boldsymbol{r}')\, S_2(\boldsymbol{r}'') \left(\left\{ - \frac{4}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{6}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} - 2 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \right.\right.\right.$$

$$\left.\left. + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \right\} I_1(x, y, \alpha) + 8 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, \alpha) \right) + \qquad (V_{5'5'})$$

$$+ C'_{v1}\, C''_{v2}\, (S_3(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}')\, S_3(\boldsymbol{r}''))\cdot \Bigg(\Bigg\{ 4 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} +$$

$$+ 2 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) + \qquad (V_{1'2''})$$

$$+ C''_{v1}\, C''_{v2}\, (S_3(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}')\, S_3(\boldsymbol{r}'')) \Bigg(\Bigg\{ - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} -$$

$$- 4 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) - 4 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) + \qquad (V_{1''2''})$$

$$+ C'_{v2}\, C'_{v3} \Bigg[(S_0(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}')\, S_0(\boldsymbol{r}'')) \Bigg(\Bigg\{ - 4 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} +$$

$$+ 2 \left(\frac{\partial}{\partial y} \frac{1}{xy} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \frac{1}{xy} \frac{\partial^2}{\partial y^2}\right) + 2 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) +$$

$$+ 4 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) + \qquad (V_{2'3'})$$

$$+ (\boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z \Bigg(\Bigg\{ - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} -$$

$$- \left(\frac{\partial}{\partial y} \frac{1}{xy} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \frac{1}{xy} \frac{\partial^2}{\partial y^2}\right) + 3 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) +$$

$$+ 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) \Bigg] +$$

$$+ C'_{v2}\, C''_{v3} \Bigg[(S_0(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}')\, S_0(\boldsymbol{r}'')) \Bigg(\Bigg\{ - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} +$$

$$+ 2 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) +$$

$$+ (\boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z \Bigg(\Bigg\{ - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy} + \qquad (V_{2'3''})$$

$$+ 2 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial x\, \partial y^2}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) +$$

$$+ C'_{v3}\, C''_{v4} \Bigg[(S_{3z}(\boldsymbol{r}')\, S_1(\boldsymbol{r}'') - S_1(\boldsymbol{r}')\, S_{3z}(\boldsymbol{r}'')) \Bigg(\Bigg\{ - 2 \left(\frac{\partial}{\partial y} \frac{1}{xy} \frac{\partial^2}{\partial x^2} +\right.$$

$$\left. + \frac{\partial}{\partial x} \frac{1}{xy} \frac{\partial^2}{\partial y^2}\right) - 2 \left(\frac{\partial^3}{\partial x^2\, \partial y} + \frac{\partial^3}{\partial y^2\, \partial x}\right) \frac{1}{xy} \Bigg\} I_1(x, y, \alpha) -$$

$$- 4 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \frac{1}{xy}\, I_2(x, y, \alpha) \Bigg) + \qquad (V_{3'4''})$$

$$+ (\boldsymbol{S}_2(\boldsymbol{r}')\times \boldsymbol{S}_0(\boldsymbol{r}'') + \boldsymbol{S}_0(\boldsymbol{r}')\times \boldsymbol{S}_2(\boldsymbol{r}''))_z \left(\left\{- 2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} + \right.\right.$$

$$\left. + \left(\frac{\partial}{\partial y}\frac{1}{xy}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\frac{1}{xy}\frac{\partial^2}{\partial y^2}\right) - 3\left(\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}\right)\frac{1}{xy}\right\} I_1(x, y, \alpha) -$$

$$\left. \left. - 2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} I_2(x, y, \alpha)\right)\right] +$$

$$+ C''_{v3} C''_{v4}\left[(S_{3z}(\boldsymbol{r}') S_1(\boldsymbol{r}'') - S_1(\boldsymbol{r}') S_{3z}(\boldsymbol{r}''))\left(+ 4\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} - \right.\right.$$

$$\left. - 2\left(\frac{\partial}{\partial y}\frac{1}{xy}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\frac{1}{xy}\frac{\partial^2}{\partial y^2}\right) + 2\left(\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}\right)\frac{1}{xy}\right) I_1(x, y, \alpha) +$$

$$+ (\boldsymbol{S}_2(\boldsymbol{r}')\times \boldsymbol{S}_0(\boldsymbol{r}'') + \boldsymbol{S}_0(\boldsymbol{r}')\times \boldsymbol{S}_2(\boldsymbol{r}''))_z \left(\left\{2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} + \right.\right. \tag{$V_{3''4''}$}$$

$$\left. + \left(\frac{\partial}{\partial y}\frac{1}{xy}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\frac{1}{xy}\frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}\right)\frac{1}{xy}\right\} I_1(x, y, \alpha) +$$

$$\left. \left. + 2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} I_2(x, y, \alpha)\right)\right] +$$

$$+ C'_{v4} C'_{v5}(S_{0z}(\boldsymbol{r}') S_2(\boldsymbol{r}'') - S_2(\boldsymbol{r}') S_{0z}(\boldsymbol{r}''))\left(\left\{- 4\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} + \right.\right. \tag{$V_{4'5'}$}$$

$$\left.\left. + 4\left(\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}\right)\frac{1}{xy}\right\} I_1(x, y, \alpha) + 4\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{1}{xy} I_2(x, y, \alpha)\right\}_{x=y=\varkappa r}.$$

We shall define ΔH as $\lim\limits_{\alpha=0} \Delta H(\alpha)$ having put the differential operators outside the integral sign, as in (57), so that the integrals left are convergent even when $\alpha = 0$. We get from (53) or (54)

$$I_1(x, y, 0) = \lim_{\alpha=0} I_1(x, y, \alpha) =$$

$$= \lim_{\alpha=0} - \frac{\partial}{\partial y}\int_0^\infty\int_0^\infty \frac{p\, d p\, d q \sin x p \cos y q}{\sqrt{p^2 + 1}\sqrt{q^2 + 1}(\sqrt{p^2 + 1} + \sqrt{q^2 + 1})} f(p, q, \alpha) \tag{58}$$

where

$$\int_0^\infty\int_0^\infty \frac{p\, d p\, d q \sin x p \cos y q}{\sqrt{p^2 + 1}\sqrt{q^2 + 1}(\sqrt{p^2 + 1} + \sqrt{q^2 + 1})} = \tag{59}$$

$$= \int_0^\infty \frac{\cos y q\, d q}{\sqrt{q^2 + 1}}\int_0^\infty \frac{p \sin x p}{p^2 - q^2} d p + \int_0^\infty \frac{p \sin x p\, d p}{\sqrt{p^2 + 1}}\int_0^\infty \frac{\cos y q\, d q}{q^2 - p^2} =$$

(taking the principal values of the integrals)

$$= \frac{\pi}{2}\int_0^\infty \frac{\cos yq\, dq}{\sqrt{q^2+1}} \cos xq - \frac{\pi}{2}\int_0^\infty \frac{p \sin xp}{\sqrt{p^2+1}}\, dp \frac{\sin yp}{p} =$$

$$= \frac{\pi}{2}\int_0^\infty \frac{\cos (x+y)\, q\, dq}{\sqrt{q^2+1}} = \frac{\pi}{2} K_0 (x+y)$$

and

$$I_2 (x, y, 0) = \lim_{\alpha=0} I_2 (x, y, \alpha) =$$

$$= \lim_{\alpha=0} \frac{\partial^3}{\partial x^2\, \partial y} \int_0^\infty \int_0^\infty \frac{dp\, dq \sin xp \cos yq}{p\,(\sqrt{p^2+1} + \sqrt{q^2+1})} f(p, q, \alpha) \qquad (60)$$

where

$$\int_0^\infty \int_0^\infty \frac{dp\, dq \sin xp \cos yq}{p\,(\sqrt{p^2+1} + \sqrt{q^2+1})} = \qquad (61)$$

$$= - \int_0^\infty \sqrt{q^2+1} \cos yq\, dq \int_0^\infty \frac{\sin xp\, dp}{p\,(p^2 - q^2)} - \int_0^\infty \frac{\sqrt{p^2+1} \sin xp\, dp}{p} \int_0^\infty \frac{\cos yq\, dq}{q^2 - p^2} =$$

(taking the principal values of the integrals)

$$= - \frac{\pi}{2} \int_0^\infty \sqrt{q^2+1} \cos yq\, dq \frac{(\cos xq - 1)}{q^2} + \frac{\pi}{2} \int_0^\infty \frac{\sqrt{p^2+1} \sin xp\, dp}{p} \cdot \frac{\sin yp}{p} =$$

$$= - \frac{\pi}{2} \left\{ \int_0^\infty \frac{\cos (x+y)\, q\, dq}{\sqrt{q^2+1}} - \int_0^\infty \frac{\cos yq\, dq}{\sqrt{q^2+1}} + \int_0^\infty \frac{(\cos (x+y)\, q - \cos yq)\, dq}{q^2 \sqrt{q^2+1}} \right\} =$$

$$= - \frac{\pi}{2} \left\{ K_0 (x+y) - K_0 (y) + F (x, y) \right\}$$

where

$$F (x, y) = \int_0^\infty \frac{(\cos (x+y)\, q - \cos yq)\, dq}{q^2 \sqrt{q^2+1}}$$

$$\frac{\partial^2 F (x, y)}{\partial x^2} = - \int_0^\infty \frac{\cos (x+y)\, q}{\sqrt{q^2+1}}\, dq = - K_0 (x+y). \qquad (62)$$

Using the well-known differential formulas for the HANKEL functions $K_0 (z)$ and $K_1 (z)$

$$\frac{\partial}{\partial x} K_0(x+y) = \frac{\partial}{\partial y} K_0(x+y) = -K_1(x+y) \tag{63}$$

$$\frac{\partial}{\partial x} K_1(x+y) = \frac{\partial}{\partial y} K_1(x+y) = -\frac{1}{x+y} K_1(x+y) - K_0(x+y) \tag{64}$$

we get from (58) and (59) resp. (60), (61) and (62)

$$I_1(x, y, 0) = \frac{\pi}{2} K_1(x+y) \tag{65}$$

$$I_2(x, y, 0) = \frac{\pi}{2}\left(\frac{\partial^2}{\partial x\, \partial y} - 1\right) K_1(x+y). \tag{66}$$

Putting $2\varkappa |\boldsymbol{r}'' - \boldsymbol{r}'| = z$ we get by (57, 63—66)

$$\Delta H_v = \frac{16}{\pi}\left(\frac{e^2}{\hbar c}\right)^2 \mu c^2 \int\int dV'\, dV'' \cdot \tag{67}$$

$$\cdot \Big\{ C'^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left[-\left(\frac{240}{z^6} + \frac{42}{z^4} + \frac{1.5}{z^2}\right) K_1(z) - \left(\frac{120}{z^5} + \frac{6}{z^3}\right) K_0(z)\right] + \qquad (V_{1'}V_{1'})$$

$$+ C''^2_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left[-\left(\frac{480}{z^6} + \frac{84}{z^4} + \frac{1.5}{z^2}\right) K_1(z) - \left(\frac{240}{z^5} + \frac{12}{z^3}\right) K_0(z)\right] + \qquad (V_{1''1''})$$

$$+ C'_{v1} C''_{v1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left[\left(\frac{36}{z^4} + \frac{3}{z^2}\right) K_1(z) + \frac{18}{z^3} K_0(z)\right] + \qquad (V_{1'1''})$$

$$+ C'^2_{v2} \Big[S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') \left(\left(\frac{320}{z^6} + \frac{68}{z^4}\right) K_1(z) + \left(\frac{160}{z^5} + \frac{14}{z^3}\right) K_0(z)\right) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') \left(-\left(\frac{400}{z^6} + \frac{82}{z^4}\right) K_1(z) - \left(\frac{200}{z^5} + \frac{14}{z^3}\right) K_0(z)\right) + \qquad (V_{2'2'})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') \left(\left(\frac{560}{z^6} + \frac{110}{z^4}\right) K_1(z) + \left(\frac{280}{z^5} + \frac{14}{z^3}\right) K_0(z)\right)\Big] +$$

$$+ C''^2_{v2} \Big[S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') \left(\left(\frac{320}{z^6} + \frac{56}{z^4}\right) K_1(z) + \left(\frac{160}{z^5} + \frac{8}{z^3}\right) K_0(z)\right) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') \left(-\left(\frac{400}{z^6} + \frac{82}{z^4}\right) K_1(z) - \left(\frac{200}{z^5} + \frac{14}{z^3}\right) K_0(z)\right) + \qquad (V_{2''2''})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') \left(\left(\frac{560}{z^6} + \frac{170}{z^4} + \frac{6}{z^2}\right) K_1(z) + \left(\frac{280}{z^5} + \frac{44}{z^3}\right) K_0(z)\right) +$$

$$+ C'^2_{v3} \Big[\boldsymbol{S}_3(\boldsymbol{r}') \boldsymbol{S}_3(\boldsymbol{r}'') \left(\left(\frac{240}{z^6} + \frac{44}{z^4}\right) K_1(z) + \left(\frac{120}{z^5} + \frac{7}{z^3}\right) K_0(z)\right) +$$

$$+ S_{3z}(\boldsymbol{r}') S_{3z}(\boldsymbol{r}'') \left(-\left(\frac{560}{z^6} + \frac{100}{z^4} + \frac{1}{z^2}\right) K_1(z) - \left(\frac{280}{z^5} + \frac{15}{z^3}\right) K_0(z)\right) + \qquad (V_{3'3'})$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}')\,\boldsymbol{S}_2(\boldsymbol{r}'')\Bigg(\Bigg(\frac{240}{z^6}+\frac{44}{z^4}+\frac{1}{z^2}\Bigg)K_1(z)+\Bigg(\frac{120}{z^5}+\frac{7}{z^3}\Bigg)K_0(z)\Bigg)+$$

$$+ S_{2z}(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{560}{z^6}+\frac{100}{z^4}+\frac{1}{z^2}\Bigg)K_1(z)-\Bigg(\frac{280}{z^5}+\frac{15}{z^3}\Bigg)K_0(z)\Bigg)\Bigg]+$$

$$+ C''^2_{v3}\Bigg[\boldsymbol{S}_3(\boldsymbol{r}')\,\boldsymbol{S}_3(\boldsymbol{r}'')\Bigg(\Bigg(\frac{100}{z^6}+\frac{24}{z^4}\Bigg)K_1(z)+\Bigg(\frac{50}{z^5}+\frac{5}{z^3}\Bigg)K_0(z)\Bigg)+$$

$$+ S_{3z}(\boldsymbol{r}')\,S_{3z}(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{140}{z^6}+\frac{40}{z^4}+\frac{1}{z^2}\Bigg)K_1(z)-\Bigg(\frac{70}{z^5}+\frac{9}{z^3}\Bigg)K_0(z)\Bigg)+$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}')\,\boldsymbol{S}_2(\boldsymbol{r}'')\Bigg(\Bigg(\frac{20}{z^6}+\frac{10}{z^4}+\frac{1}{z^2}\Bigg)K_1(z)+\Bigg(\frac{10}{z^4}+\frac{3}{z^3}\Bigg)K_0(z)\Bigg)+ \qquad (V_{3''3''})$$

$$+ S_{2z}(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{140}{z^6}+\frac{40}{z^4}+\frac{1}{z^2}\Bigg)K_1(z)-\Bigg(\frac{70}{z^5}+\frac{9}{z^3}\Bigg)K_0(z)\Bigg)\Bigg]+$$

$$+ C'_{v3}\,C''_{v3}\Bigg[\boldsymbol{S}_3(\boldsymbol{r}')\,\boldsymbol{S}_3(\boldsymbol{r}'')\Bigg(\frac{4}{z^4}K_1(z)+\frac{2}{z^3}K_0(z)\Bigg)+$$

$$+ S_{3z}(\boldsymbol{r}')\,S_{3z}(\boldsymbol{r}'')\Bigg(\Bigg(\frac{20}{z^4}+\frac{2}{z^2}\Bigg)K_1(z)+\frac{10}{z^3}K_0(z)\Bigg)+ \qquad (V_{3'3''})$$

$$+ (\boldsymbol{S}_2(\boldsymbol{r}')\,\boldsymbol{S}_2(\boldsymbol{r}'')-S_{2z}(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}''))\Bigg(-\Bigg(\frac{20}{z^4}+\frac{2}{z^2}\Bigg)K_1(z)-\frac{10}{z^3}K_0(z)\Bigg)\Bigg]+$$

$$+ C'^2_{v4}\Bigg[\boldsymbol{S}_0(\boldsymbol{r}')\,\boldsymbol{S}_0(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{400}{z^6}+\frac{52}{z^4}\Bigg)K_1(z)-\frac{200}{z^5}K_0(z)\Bigg)+$$

$$+ S_{0z}(\boldsymbol{r}')\,S_{0z}(\boldsymbol{r}'')\Bigg(\Bigg(\frac{560}{z^6}+\frac{100}{z^4}\Bigg)K_1(z)+\Bigg(\frac{280}{z^5}+\frac{12}{z^3}\Bigg)K_0(z)\Bigg)+ \qquad (V_{4'4'})$$

$$+ S_1(\boldsymbol{r}')\,S_1(\boldsymbol{r}'')\Bigg(\Bigg(\frac{320}{z^6}+\frac{40}{z^4}\Bigg)K_1(z)+\frac{160}{z^5}K_0(z)\Bigg)\Bigg]+$$

$$+ C''^2_{v4}\Bigg[\boldsymbol{S}_0(\boldsymbol{r}')\,\boldsymbol{S}_0(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{400}{z^6}+\frac{100}{z^4}+\frac{4}{z^2}\Bigg)K_1(z)-\Bigg(\frac{200}{z^5}+\frac{24}{z^3}\Bigg)K_0(z)\Bigg)+$$

$$+ S_{0z}(\boldsymbol{r}')\,S_{0z}(\boldsymbol{r}'')\Bigg(\Bigg(\frac{560}{z^6}+\frac{140}{z^4}+\frac{4}{z^2}\Bigg)K_1(z)+\Bigg(\frac{280}{z^5}+\frac{32}{z^3}\Bigg)K_0(z)\Bigg)+ \qquad (V_{4''4''})$$

$$+ S_1(\boldsymbol{r}')\,S_1(\boldsymbol{r}'')\Bigg(\Bigg(\frac{320}{z^6}+\frac{80}{z^4}+\frac{4}{z^2}\Bigg)K_1(z)+\Bigg(\frac{160}{z^5}+\frac{20}{z^3}\Bigg)K_0(z)\Bigg)\Bigg]+$$

$$+ C'^2_{v5}\,S_2(\boldsymbol{r}')\,S_2(\boldsymbol{r}'')\Bigg(-\Bigg(\frac{480}{z^6}+\frac{84}{z^4}\Bigg)K_1(z)-\Bigg(\frac{240}{z^5}+\frac{12}{z^3}\Bigg)K_0(z)\Bigg)+ \qquad (V_{5'5'})$$

$$+ C'_{v1}\,C''_{v2}\,(S_3(\boldsymbol{r}')\,\boldsymbol{S}_1(\boldsymbol{r}'')-\boldsymbol{S}_1(\boldsymbol{r}')\,S_3(\boldsymbol{r}''))_z\Bigg[\Bigg(\frac{60}{z^5}+\frac{15}{z^3}\Bigg)K_1(z)+$$

$$+\Bigg(\frac{30}{z^4}+\frac{3}{z^2}\Bigg)K_0(z)\Bigg]+ \qquad (V_{1'2''})$$

$$+ C''_{v1}\, C''_{v2}\, (S_3(\boldsymbol{r}')\, \boldsymbol{S}_1(\boldsymbol{r}'') - \boldsymbol{S}_1(\boldsymbol{r}')\, S_3(\boldsymbol{r}''))_z \left[- \left(\frac{120}{z^5} + \frac{21}{z^3} \right) K_1(z) - \right.$$

$$\left. - \left(\frac{60}{z^4} + \frac{3}{z^2} \right) K_0(z) \right] + \qquad (V_{1''2''})$$

$$+ C'_{v2}\, C'_{v3}\, (S_0(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'') - \boldsymbol{S}_2(\boldsymbol{r}')\, S_0(\boldsymbol{r}'') + \boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z \cdot$$

$$\cdot \left[\left(\frac{80}{z^5} + \frac{8}{z^3} \right) K_1(z) + \frac{40}{z^4} K_0(z) \right] + \qquad (V_{2'3'})$$

$$+ C'_{v2}\, C''_{v3}\, (S_0(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'') - \boldsymbol{S}_2(\boldsymbol{r}')\, S_0(\boldsymbol{r}'') + \boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z \cdot$$

$$\cdot \left[\left(\frac{60}{z^5} + \frac{6}{z^3} \right) K_1(z) + \frac{30}{z^4} K_0(z) \right] + \qquad (V_{2'3''})$$

$$+ C'_{v3}\, C''_{v4}\, (\boldsymbol{S}_3(\boldsymbol{r}')\, S_1(\boldsymbol{r}'') - S_1(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_2(\boldsymbol{r}') \times \boldsymbol{S}_0(\boldsymbol{r}'') + \boldsymbol{S}_0(\boldsymbol{r}') \times \boldsymbol{S}_2(\boldsymbol{r}''))_z \cdot$$

$$\cdot \left[- \left(\frac{80}{z^5} + \frac{14}{z^3} \right) K_1(z) - \left(\frac{40}{z^4} + \frac{2}{z^2} \right) K_0(z) \right] + \qquad (V_{3'4''})$$

$$+ C''_{v3}\, C''_{v4}\, (\boldsymbol{S}_3(\boldsymbol{r}')\, S_1(\boldsymbol{r}'') - S_1(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_2(\boldsymbol{r}') \times \boldsymbol{S}_0(\boldsymbol{r}'') + \boldsymbol{S}_0(\boldsymbol{r}') \times \boldsymbol{S}_2(\boldsymbol{r}''))_z \cdot$$

$$\cdot \left[\left(\frac{40}{z^5} + \frac{10}{z^3} \right) K_1(z) + \left(\frac{20}{z^4} + \frac{2}{z^2} \right) K_0(z) \right] + \qquad (V_{3''4''})$$

$$+ C'_{v4}\, C'_{v5}\, (\boldsymbol{S}_0(\boldsymbol{r}')\, S_2(\boldsymbol{r}'') - S_2(\boldsymbol{r}')\, \boldsymbol{S}_0(\boldsymbol{r}''))_z \left[\left(\frac{120}{z^5} + \frac{12}{z^3} \right) K_1(z) + \frac{60}{z^4} K_0(z) \right] \Bigg\}. \qquad (V_{4'5'})$$

B. Scalar Mesons

The calculation is in this case analogous to the vector case. The equations (46) are still valid, but the indices $r'\, r''\, s'\, s''$ disappear. The same coordinate systems are introduced where $\boldsymbol{k}$ is given from the third equation in (48). Thus we get from (37)

$$\Delta H_s(a) = - \frac{1}{2\pi^2} \left(\frac{e^2}{\hbar c} \right)^2 \mu c^2 \int_0^\infty \int_0^\infty d\,V'\, d\,V'' \int_0^\infty \int_0^\infty d\,p\, d\,q \cdot \qquad (68)$$

$$\cdot \frac{f(p, q, a)\, p^2 q^2}{\sqrt{p^2+1}\, \sqrt{q^2+1}\, (\sqrt{p^2+1} + \sqrt{q^2+1})} \int_{-1}^{+1} \int_{-1}^{+1} e^{i x r (p u - q v)}\, d\,u\, d\,v \cdot$$

$$\cdot \{ C'^2_{s1}\, S_3(\boldsymbol{r}')\, S_3(\boldsymbol{r}'') \cdot 2 + \qquad (S_{1'1'})$$

$$+ C''^2_{s1}\, S_3(\boldsymbol{r}')\, S_3(\boldsymbol{r}'')\, (2 + 2\,(p^2 + q^2) + 3\,p^2 q^2 - p^2 q^2 (u^2 + v^2) +$$

$$+ 3\,p^2 q^2 u^2 v^2 + 4\,p\,q\,u\,v \sqrt{p^2+1} \sqrt{q^2+1}) + \qquad (S_{1''1''})$$

$$+ C'_{s1}\, C''_{s1}\, S_3(\boldsymbol{r}')\, S_3(\boldsymbol{r}'')\, (4\,p\,q\,u\,v + 4 \sqrt{p^2+1} \sqrt{q^2+1}) + \qquad (S_{1'1''})$$

$$+ C_{s2}^{'2} [S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') (4 + 2(p^2 + q^2) - 4 \sqrt{p^2 + 1} \sqrt{q^2 + 1}) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') (p^2 + q^2 - (p^2 u^2 + q^2 v^2)) + \qquad (S_{2'2'})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') (-(p^2 + q^2) + 3(p^2 u^2 + q^2 v^2) + 4 p q u v)] +$$

$$+ C_{s2}^{''2} [S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') (4 + 2(p^2 + q^2) + 4 \sqrt{p^2 + 1} \sqrt{q^2 + 1}) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') (p^2 + q^2 - (p^2 u^2 + q^2 v^2)) + \qquad (S_{2''2''})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') (-(p^2 + q^2) + 3(p^2 u^2 + q^2 v^2) - 4 p q u v)] +$$

$$+ C_{s3}^{'2} [\boldsymbol{S}_3(\boldsymbol{r}') \boldsymbol{S}_3(\boldsymbol{r}'') (p^2 q^2 (u^2 + v^2) - 2 p^2 q^2 u^2 v^2) +$$

$$+ S_{3z}(\boldsymbol{r}') S_{3z}(\boldsymbol{r}'') (p^2 q^2 - 2 p^2 q^2 (u^2 + v^2) + 3 p^2 q^2 u^2 v^2) +$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}') \boldsymbol{S}_2(\boldsymbol{r}'') (p^2 + q^2 - (p^2 u^2 + q^2 v^2) + 2 p^2 q^2 - p^2 q^2 (u^2 + v^2)) + \qquad (S_{3'3'})$$

$$+ S_{2z}(\boldsymbol{r}') S_{2z}(\boldsymbol{r}'') (-(p^2 + q^2) + 3(p^2 u^2 + q^2 v^2) - 2 p^2 q^2 +$$

$$+ 3 p^2 q^2 (u^2 + v^2) + 4 p q u v \sqrt{p^2 + 1} \sqrt{q^2 + 1})] +$$

$$+ i C_{s1}' C_{s2}'' (S_3(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}') S_3(\boldsymbol{r}'')) (2(p u - q v)) + \qquad (S_{1'} S_{2''})$$

$$+ i C_{s1}'' C_{s2}'' (S_3(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}') S_3(\boldsymbol{r}'')) (2 p q u v)(p u - q v) + \qquad (S_{1''} S_{2''})$$

$$+ 2(p u - q v) \sqrt{p^2 + 1} \sqrt{q^2 + 1}) +$$

$$+ i C_{s2}' C_{s3}' [(S_0(\boldsymbol{r}') S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}') S_0(\boldsymbol{r}'')) (2(p u - q v) -$$

$$- 2(p^2 q v - p q^2 u) - 2(p u - q v) \sqrt{p^2 + 1} \sqrt{q^2 + 1}) +$$

$$+ (\boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z (p^2 q v - \qquad (S_{2'3'})$$

$$- p q^2 u - p q u v (p u - q v))]\}.$$

Similarly to (57) we get

$$\Delta H_s(a) = -\frac{2}{\pi^2} \left(\frac{e^2}{\hbar c}\right) \mu c^2 \int\int d V' \, d V'' \cdot \tag{69}$$

$$\cdot \Big\{ C_{s1}^{'2} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \frac{2}{xy} I_1(x, y, a) + \qquad (S_{1'1'})$$

$$+ C_{s1}^{''2} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \Big(\Big\{ \frac{2}{xy} - \frac{2}{xy} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{3}{xy} \frac{\partial^4}{\partial x^2 \partial y^2} -$$

$$- \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) + 3 \frac{\partial^4}{\partial x^2 \partial y^2} \frac{1}{xy} \Big\} I_1(x, y, a) + \qquad (S_{1''1''})$$

$$+ 4 \frac{\partial^2}{\partial x \partial y} \frac{1}{xy} I_2(x, y, a) \Big) +$$

$$+ C'_{s1}\, C''_{s1}\, S_3(\boldsymbol{r}')\, S_3(\boldsymbol{r}'') \Biggl(4 \frac{\partial^2}{\partial x\, \partial y} \frac{1}{x y} I_1(x, y, a) + \frac{4}{x y} I_2(x, y, a)\Biggr) + \qquad (S_{1'1''})$$

$$+ C'^2_{s2} \Biggl[S_0(\boldsymbol{r}')\, S_0(\boldsymbol{r}'') \Biggl(\Biggl\{ \frac{4}{x y} - \frac{2}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \Biggr\} I_1(x, y, a) -$$

$$- \frac{4}{x y} I_2(x, y, a) \Biggr) + \qquad (S_{2'2'})$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}')\, \boldsymbol{S}_1(\boldsymbol{r}'') \Biggl(- \frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) + \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} \Biggr) I_1(x, y, a) +$$

$$+ S_{1z}(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'') \Biggl(\frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) - 3 \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} +$$

$$+ 4 \frac{\partial^2}{\partial x\, \partial y} \frac{1}{x y} \Biggr) I_1(x, y, a) \Biggr] +$$

$$+ C''^2_{s2} \Biggl[S_0(\boldsymbol{r}')\, S_0(\boldsymbol{r}'') \Biggl(\Biggl\{ \frac{4}{x y} - \frac{2}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \Biggr\} I_1(x, y, a) +$$

$$+ \frac{4}{x y} I_2(x, y, a) \Biggr) + \qquad (S_{2''2''})$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}')\, \boldsymbol{S}_1(\boldsymbol{r}'') \Biggl(- \frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) + \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} \Biggr) I_1(x, y, a) +$$

$$+ S_{1z}(\boldsymbol{r}')\, S_{1z}(\boldsymbol{r}'') \Biggl(\frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{d y^2} \Biggr) - 3 \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} -$$

$$- 4 \frac{\partial^2}{\partial x\, \partial y} \frac{1}{x y} \Biggr) I_1(x, y, a) \Biggr] +$$

$$+ C'^2_{s3} \Biggl[\boldsymbol{S}_3(\boldsymbol{r}')\, \boldsymbol{S}_3(\boldsymbol{r}'') \Biggl(\frac{\partial^2}{\partial x^2} \frac{1}{x y} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{x y} \frac{\partial^2}{\partial x^2} - 2 \frac{\partial^4}{\partial x^2\, \partial y^2} \frac{1}{x y} \Biggr) I_1(x, y. a) +$$

$$+ S_{3z}(\boldsymbol{r}')\, S_{3z}(\boldsymbol{r}'') \Biggl(\frac{1}{x y} \frac{\partial^4}{\partial x^2\, \partial y^2} - 2 \Biggl(\frac{\partial^2}{\partial x^2} \frac{1}{x y} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{x y} \frac{\partial^2}{\partial x^2} \Biggr) +$$

$$+ 3 \frac{\partial^4}{\partial x^2\, \partial y^2} \frac{1}{x y} \Biggr) I_1(x, y, a) +$$

$$+ \boldsymbol{S}_2(\boldsymbol{r}')\, \boldsymbol{S}_2(\boldsymbol{r}'') \Biggl(- \frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) + \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} + \frac{2}{x y} \frac{\partial^4}{\partial x^2\, \partial y^2} - \qquad (S_{3'3'})$$

$$- \frac{\partial^2}{\partial x^2} \frac{1}{x y} \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^2} \frac{1}{x y} \frac{\partial^2}{\partial x^2} \Biggr) I_1(x, y, a) +$$

$$+ S_{2z}(\boldsymbol{r}')\, S_{2z}(\boldsymbol{r}'') \Biggl(\Biggl\{ + \frac{1}{x y} \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) - 3 \Biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Biggr) \frac{1}{x y} - \frac{2}{x y} \frac{\partial^4}{\partial x^2\, \partial y^2} +$$

$$- 3 \left(\frac{\partial^2}{\partial x^2} \frac{1}{xy} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \frac{1}{xy} \frac{\partial^2}{\partial x^2} \right) \Bigg\} I_1 (x, y, \alpha) + 4 \frac{\partial^2}{\partial x \, \partial y} \frac{1}{xy} I_2 (x, y, \alpha) \Bigg) +$$

$$+ C'_{s1} C''_{s2} (S_3(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}') S_3(\boldsymbol{r}'')) \left(2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{1}{xy} I_1 (x, y, \alpha) \right) + \qquad (S_{1'2''})$$

$$+ C''_{s1} C''_{s2} (S_3(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') - S_{1z}(\boldsymbol{r}') S_3(\boldsymbol{r}'')) \Bigg(2 \Bigg(\frac{\partial^3}{\partial x^2 \, \partial y} + \qquad (S_{1''2''})$$

$$+ \frac{\partial^3}{\partial x \, \partial y^2} \Bigg) \frac{1}{xy} I_1 (x, y, \alpha) + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{1}{xy} I_2 (x, y, \alpha) \Bigg) +$$

$$+ C'_{s2} C'_{s3} \Bigg[(S_0(\boldsymbol{r}') S_{2z}(\boldsymbol{r}'') - S_{2z}(\boldsymbol{r}') S_0(\boldsymbol{r}'')) \Bigg(\Bigg\{ 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{1}{xy} -$$

$$- 2 \left(\frac{\partial}{\partial y} \frac{1}{xy} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \frac{1}{xy} \frac{\partial^2}{\partial y^2} \right) \Bigg\} I_1 (x, y, \alpha) - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{1}{xy} I_2 (x, y, \alpha) \Bigg) + \qquad (S_{2'3'})$$

$$+ (\boldsymbol{S}_1(\boldsymbol{r}') \times \boldsymbol{S}_3(\boldsymbol{r}'') + \boldsymbol{S}_3(\boldsymbol{r}') \times \boldsymbol{S}_1(\boldsymbol{r}''))_z \Bigg(\frac{\partial}{\partial y} \frac{1}{xy} \frac{\partial^2}{\partial x^2} +$$

$$+ \frac{\partial}{\partial x} \frac{1}{xy} \frac{\partial^2}{\partial y^2} - \left(\frac{\partial^3}{\partial x^2 \, \partial y} + \frac{\partial^3}{\partial x \, \partial y^2} \right) \frac{1}{xy} \Bigg) I_1 (x, y, \alpha) \Bigg] \Bigg\}$$

where $x = y = \varkappa |\boldsymbol{r}'' - \boldsymbol{r}'|$ and all other interaction terms are zero. Putting $\alpha = 0$ we get (cf. (67))

$$\varDelta H_s = \frac{16}{\pi} \left(\frac{e^2}{\hbar c} \right)^2 \mu c^2 \int \int d V' \, d V'' \cdot \qquad (70)$$

$$\cdot \Bigg\{ C'^2_{s1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left(- \frac{0.5}{z^2} K_1(z) \right) + \qquad (S_{1'1'})$$

$$+ C''^2_{s1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left(- \left(\frac{240}{z^6} + \frac{42}{z^4} + \frac{0.5}{z^2} \right) K_1(z) - \left(\frac{120}{z^5} + \frac{6}{z^3} \right) K_0(z) \right) + \qquad (S_{1''1''})$$

$$+ C'_{s1} C''_{s1} S_3(\boldsymbol{r}') S_3(\boldsymbol{r}'') \left(- \left(\frac{12}{z^4} + \frac{1}{z^2} \right) K_1(z) - \frac{6}{z^3} K_0(z) \right) + \qquad (S_{1'1''})$$

$$+ C'^2_{s2} \Bigg[S_0(\boldsymbol{r}') S_0(\boldsymbol{r}'') \left(\frac{4}{z^4} K_1(z) + \frac{2}{z^3} K_0(z) \right) +$$

$$+ \boldsymbol{S}_1(\boldsymbol{r}') \boldsymbol{S}_1(\boldsymbol{r}'') \left(- \frac{6}{z^4} K_1(z) - \frac{2}{z^3} K_0(z) \right) + \qquad (S_{2'2'})$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') \left(\frac{10}{z^4} K_1(z) + \frac{2}{z^3} K_0(z) \right) \Bigg] +$$

$$+ C''^2_{s2} \Bigg[\boldsymbol{S}_1(\boldsymbol{r}') \, \boldsymbol{S}_1(\boldsymbol{r}'') \left(- \frac{6}{z^4} K_1(z) - \frac{2}{z^3} K_0(z) \right) +$$

$$+ S_{1z}(\boldsymbol{r}') S_{1z}(\boldsymbol{r}'') \left(\left(\frac{30}{z^4} + \frac{2}{z^2} \right) K_1(z) + \frac{12}{z^3} K_0(z) \right) \Bigg] + \qquad (S_{2''2''})$$

$$\begin{aligned}
&+ C_{s3}^{2}\left[\boldsymbol{S}_3(\boldsymbol{r}')\,\boldsymbol{S}_3(\boldsymbol{r}'')\left(\left(\frac{100}{z^6}+\frac{16}{z^4}\right)K_1(z)+\left(\frac{50}{z^5}+\frac{2}{z^3}\right)K_0(z)\right)+\right.\\
&+ S_{3z}(\boldsymbol{r}')\,S_{3z}(\boldsymbol{r}'')\left(-\left(\frac{140}{z^6}+\frac{20}{z^4}\right)K_1(z)-\left(\frac{70}{z^5}+\frac{2}{z^3}\right)K_0(z)\right)+ \qquad (S_{3'3'})\\
&+ \boldsymbol{S}_2(\boldsymbol{r}')\,\boldsymbol{S}_2(\boldsymbol{r}'')\left(\left(\frac{20}{z^6}+\frac{2}{z^4}\right)K_1(z)+\frac{10}{z^5}K_0(z)\right)+\\
&\left.+ S_{2z}(\boldsymbol{r}')\,S_{2z}(\boldsymbol{r}'')\left(-\left(\frac{140}{z^6}+\frac{20}{z^4}\right)K_1(z)-\left(\frac{70}{z^5}+\frac{2}{z^5}\right)K_0(z)\right)\right]+\\
&+ C'_{s1}\,C''_{s2}\,(S_3(\boldsymbol{r}')\,\boldsymbol{S}_1(\boldsymbol{r}'')-\boldsymbol{S}_1(\boldsymbol{r}')\,S_3(\boldsymbol{r}''))_z\left(\frac{3}{z^3}K_1(z)+\frac{1}{z^2}K_0(z)\right)+ \qquad (S_{1'2''})\\
&+ C''_{s1}\,C''_{s2}\,(S_3(\boldsymbol{r}')\,\boldsymbol{S}_1(\boldsymbol{r}'')-\boldsymbol{S}_1(\boldsymbol{r}')\,S_3(\boldsymbol{r}''))_z\left(\left(\frac{60}{z^5}+\frac{9}{z^3}\right)K_1(z)+\right.\\
&\qquad\qquad\left.+\left(\frac{30}{z^4}+\frac{1}{z^2}\right)K_0(z)\right)+ \qquad (S_{1''2''})\\
&+ C'_{s2}\,C'_{s3}\,(S_0(\boldsymbol{r}')\,\boldsymbol{S}_2(\boldsymbol{r}'')-\boldsymbol{S}_2(\boldsymbol{r}')\,S_0(\boldsymbol{r}'')+\boldsymbol{S}_1(\boldsymbol{r}')\times\boldsymbol{S}_3(\boldsymbol{r}'')+\boldsymbol{S}_3(\boldsymbol{r}')\times\boldsymbol{S}_1(\boldsymbol{r}''))_z\,\cdot\\
&\qquad\qquad\left.\cdot\left(-\left(\frac{20}{z^5}+\frac{2}{z^3}\right)K_1(z)-\frac{10}{z^4}K_0(z)\right)\right\}
\end{aligned}$$

where $z = 2\varkappa\,|\boldsymbol{r}'' - \boldsymbol{r}'|$.

Chapter II

On the question of convergence in the meson pair theory

§ 1. A rigorous determination of the total field energy in some particular cases

As is well known, we get an infinite contribution to the meson field, if, assuming pair interaction between the source and the field, we introduce a point source in the meson field. The interaction energy between two point sources is infinite too. Consequently, it is necessary to make another assumption as to the shape of the source. We might assume a source of finite extension. Such a source could for example give a rough picture of a nucleon that is, as is suggested by KLEIN [48], assumed to be built up from several smaller particles which are the real sources of the field. Our assumption is not, of course, relativistically invariant and we shall only use it when the source is at rest.[1] Chapter I can immediately be applied to a source function of this kind. Assuming this source, we want to investigate the self-energy of the source and the interaction energy between two sources. We shall at first use a rigorous method restricting ourselves to a few particular cases.

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[1] This is not the assumption of an extended source usually made in the pair theory. (See e. g. WENTZEL [41, 42], PAULI and HU [45]).

In the lowest state (the state of »no mesons») the quantization of the field gives a total field energy equal to half the sum of the moduli of all the energy values which are obtained from the energy states of the field equation. As is shown by HJALMARS [49], this is true not only if we have an unperturbed meson field but also if we have an interaction with a source which is bilinear in the meson field (pair interaction). In order to obtain these energy values we shall imagine our system of field and source included in some space of finite but large dimensions, requiring our field function to be zero on the surface of the space. Thus we get a discontinuous energy spectrum. We sum the moduli of the energy values letting the dimensions of the space become infinitely large. In doing this we shall bear in mind which terms of the energy belong to the unperturbed system, which terms give the self-energy of the single sources and which terms give the interaction energy between the sources.

We shall only consider scalar mesons and we shall assume that the contribution to the HAMILTONIAN from the interaction between the source and the meson field is

$$\Delta H = \frac{\gamma}{8\pi} \int U^* U \,\Omega \,d V \tag{1}$$

where U is the meson field, and $\gamma\,\Omega$ is the source function.[1] The integrals must be taken over the space, defined above. Of course, $\int \Omega \,d V$ ought to be finite. We shall, however, discuss some cases where this condition is not fulfilled as they are rather simple and give some interesting features of the theory.

At first we shall determine the energy states of the equation of the meson field. This equation is obtained from the HAMILTONIAN in the usual way.[2] The time independent equation is

$$(\Delta + k^2 - \gamma\,\Omega)\, U(x, y, z) = 0$$

where $k = \sqrt{\frac{E^2}{(\hbar c)^2} - \varkappa^2}$ and E is the energy of the state. We assume that the source function is spherically symmetrical and that, if we consider two sources, the distance between their centres is zero. Then we can introduce spherical polar coordinates r, ϑ, φ, taking the centre of the source as the origin of the coordinate system. Separating the coordinates we have the well-known solution

$$U = \frac{1}{r}\, U_l(r)\, Y_{lm}(\vartheta, \varphi).$$

$U_l(r)$ is determined by

$$\left(\frac{d^2}{d r^2} + k^2 - \frac{l(l+1)}{r^2} - \gamma\,\Omega(r)\right) U_l(r) = 0. \tag{2}$$

---

[1] WENTZEL assumes $\Delta H = \gamma \int U^* \Omega \,d V \int U \,\Omega \,d V$ and PAULI and HU assume expressions corresponding to this one.

[2] Generally the field equation ought to be deduced from the DIRAC equation and not from the HAMILTONIAN in Chapter I, as this HAMILTONIAN is usually not correct except to the second approximation (cf. § 2 of this chapter). In the particular case here, the HAMILTONIAN is quite correct.

Corresponding to a given value of l we have $2l+1$ values of m, i. e. the corresponding energy state shows a $(2l+1)$: fold degeneracy. Now we imagine the whole system included in a sphere with the same centre as the source, requiring U to be finite, continuous and to have a continuous derivative inside the sphere and to be zero on the surface of the sphere. Thus U must be $=0$, when $r=0$. The radius R of the sphere may be considered to be large. Summing the moduli of the energies, we let the radius of the sphere tend to infinity. Although the total field energy is a function of this radius, the interaction energy is independent of it.

If $\Omega \equiv 0$ the solution which is $=0$, when $r=0$, is

$$U_0(R) = c\sqrt{k_0 r}\, J_{l+\frac{1}{2}}(k_0 r)$$

where c is a normalization constant and $J_{l+\frac{1}{2}}(k_0 r)$ is an ordinary BESSEL function. When $r=R$

$$U_0(R) = c\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin\left(k_0 R - \tfrac{1}{2} l \pi\right) + 0\left(\frac{1}{R}\right) = 0 \cdot \tag{3}$$

Neglecting terms of the order of magnitude of $\frac{1}{R}$, this gives

$$k_0 R - \tfrac{1}{2} l \pi = n \pi \tag{4}$$

where n is any integer such that k_0 is positive. We have

$$E_{0k} = \pm\, \hbar c \sqrt{k_0^2 + \varkappa^2}.$$

As we are only interested in the modulus we may drop the minus sign. If $\Omega \neq 0$ and if it tends to zero swiftly enough when r tends to infinity we have approximately BESSEL's equation for large r, i. e.

$$U(R) \sim \sin\left(k R - \tfrac{1}{2} l \pi + \delta(k, l, \gamma)\right) + 0\left(\frac{1}{R}\right) \tag{5}$$

where the phase shift $\delta(k, l, \gamma)$ is determined by the particular choise of Ω.

Neglecting as before, $U(R)=0$ gives

$$k R - \tfrac{1}{2} l \pi + \delta(k, l, \gamma) = n \pi = k_0 R - \tfrac{1}{2} l \pi \tag{6}$$

or

$$k = k_0 - \frac{\delta(k, l, \gamma)}{R}. \tag{7}$$

Dropping the minus sign as above, we get

$$E_k = \hbar c \sqrt{k^2 + \varkappa^2} = \hbar c \sqrt{k_0^2 + \varkappa^2 - 2 k_0 \frac{\delta(k_0, l, \gamma)}{R}}$$

where we have neglected terms of the order of magnitude of $\frac{1}{R^2}$ under the radical sign.

$$E_k = \hbar c \left\{ \sqrt{k_0^2 + \varkappa^2} - \frac{1}{R} \frac{k_0}{\sqrt{k_0^2 + \varkappa^2}} \delta(k_0, l, \gamma) + 0\left(\frac{1}{R^2}\right) \right\}. \tag{8}$$

The total self-energy of the meson field is

$$E_{0R} = \sum_{l=0}^{\infty} (2l+1) \sum_{k_0} E_{k_0}$$

where the sum has to be taken over every value of k_0 determined in (4). The total energy of the whole system is

$$E_R = \sum_{l=0}^{\infty} (2l+1) \sum_{k} E_k = \sum_{l=0}^{\infty} (2l+1) \sum_{k_0} E_k$$

where k is defined as a function of k_0 in (7). If we keep l constant we can put $\varDelta k_0 \cdot \frac{R}{\pi} = 1$ from (4). Thus the interaction energy between the source and the field

$$\varDelta E_R = E_R - E_{0R} = \frac{R}{\pi} \sum_{l=0}^{\infty} (2l+1) \sum_{k_0} (E_k - E_{k_0}) \varDelta k_0 =$$

$$= -\frac{\hbar c}{\pi} \sum_{l=0}^{\infty} (2l+1) \sum_{k_0} \frac{k_0}{\sqrt{k_0^2 + \varkappa^2}} \delta(k_0, l, \gamma) \varDelta k_0$$

and in the limit of an infinite radius of the sphere

$$\varDelta E = \lim_{R=\infty} \varDelta E_R = -\frac{\hbar c}{\pi} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} \frac{k_0}{\sqrt{k_0^2 + \varkappa^2}} \delta(k_0, l, \gamma)\, d k_0. \tag{9}$$

We shall discuss three cases: $\Omega = \frac{1}{r}$ (A), $\Omega = \frac{1}{r^2}$ (B), and $\Omega =$ constant, $r <$ a given radius a and $\Omega = 0$, $r > a$ (C). In the two first cases, however, $\int \Omega\, d V$ is not finite. The solution of (2) when $r = R$ is in (A) not exactly of the type (5) and we cannot determine the self-energy in this case.

We shall determine the self-energy of the source $\varDelta E(\gamma)$ and the interaction energy between two identical, coincident sources. We get this interaction energy if we subtract the self-energy $(2 \varDelta E(\gamma))$ of two sources, both with the interaction constant γ, considering their distance to be infinite, from the self-energy $\varDelta E(2\gamma)$ of a source with the interaction constant 2γ — which source we get if the two sources coincide. The linear terms in γ disappear and therefore we are particularly interested in the higher terms.

(A) If we put $\Omega = \frac{1}{r}$ we get

$$\left(\frac{d^2}{d r^2} + k^2 - \gamma \frac{1}{r} - \frac{l(l+1)}{r^2} \right) U(r) = 0. \tag{2 A}$$

The solution which is $= 0$ at the origin is given by the confluent hypergeometric function[1] $M_{\frac{i}{2k}\gamma,\, l+\frac{1}{2}}(2\,k\,i\,r)$ and for $r = R$ we get

$$U(R) \sim \sin\left(k\,R - \frac{\gamma}{2\,k}\log 2\,k\,R - l\frac{\pi}{2} + \delta(k, l, \gamma)\right) \tag{5 A}$$

where

$$\delta(k, l, \gamma) = \arg\Gamma\left(l + 1 + i\frac{\gamma}{2\,k}\right).$$

We have

$$k \approx k_0 + \frac{\gamma}{2\,k_0}\,\frac{\log 2\,k_0\,R}{R} - \frac{\delta(k_0, l, \gamma)}{R}.$$

In the expressions for the self-energy of the source the term containing $\log R$ tends to infinity with R. This term, however, is only linear in γ and does not give any contribution to the interaction energy. Omitting this term we have (9)

$$\frac{\pi}{c\,\hbar}\cdot\Delta E = -\sum_{0}^{\infty}(2\,l + 1)\int_{0}^{\infty}\frac{k_0}{\sqrt{k_0^2 + \varkappa^2}}\arg\Gamma\left(l + 1 + i\frac{\gamma}{2\,k_0}\right)d\,k_0.$$

We can expand[2]

$$\arg\Gamma\left(l + 1 + i\frac{\gamma}{2\,k_0}\right) = \sum_{n=1}^{\infty}\left(\frac{\frac{\gamma}{2\,k_0}}{l + n} - \operatorname{arctg}\frac{\frac{\gamma}{2\,k_0}}{l + n}\right) + \frac{\gamma}{2\,k_0}\,\psi(l + 1).$$

Omitting some terms linear in γ, by an elementary integration we get

$$\frac{\pi}{c\,\hbar}\,\Delta E = \sum_{l=0}^{\infty}\sum_{n=1}^{\infty}(2\,l + 1)\left\{\frac{\gamma}{2\,(l+n)} + \sqrt{\varkappa^2 - \left(\frac{\gamma}{2\,(l+n)}\right)^2}\arccos\frac{1}{2\,\varkappa}\,\frac{\gamma}{l+n} - \frac{\pi}{2}\varkappa\right\}.$$

For $l > L$ and $n > N$ (L and N large) we can expand the terms in the series in a Taylor series

$$\Delta E = \sum_{l=0}^{L}\sum_{n=1}^{N}(2\,l+1)\left\{\frac{\gamma}{2\,(l+n)} + \sqrt{\varkappa^2 - \left(\frac{\gamma}{2\,(l+n)}\right)^2}\arccos\frac{1}{2\,\varkappa}\,\frac{\gamma}{l+n} - \frac{\pi}{2}\varkappa\right\} +$$

$$+ \sum_{l=L}^{\infty}\sum_{n=N}^{\infty}\left\{-\frac{\pi}{4\,\varkappa}(2\,l + 1)\left(\frac{\gamma}{2\,(l + n)}\right)^2 + \frac{1}{3\,\varkappa}(2\,l + 1)\left(\frac{\gamma}{2\,(l + n)}\right)^3 + \cdots\right\}$$

but

$$\sum_{l=L}^{\infty}\sum_{n=N}^{\infty}\frac{2\,l + 1}{(l + n)^2} \approx \sum_{l=L}^{\infty}\frac{2\,l + 1}{l + N}.$$

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[1] See e. g. WHITTAKER-WATSON, *Modern Analysis* 1940 Ch. XVI.
[2] NIELSEN, *Theorie der Gammafunktion* 1906 p. 23.

Thus the double series does not converge. In this case we do not only get an infinite self-energy (of which we cannot have any explicit expression) but an infinite interaction energy too, the divergency of which is linear.

(B). If we put $\Omega = \frac{1}{r^2}$ we get

$$\left(\frac{d^2}{d\,r^2} + k^2 - \frac{\gamma + l\,(l+1)}{r^2}\right) U(r) = 0. \tag{2 B}$$

Similarly to (2) its solutions are BESSEL functions the order of which are

$$l' + \tfrac{1}{2} = (l + \tfrac{1}{2}) \sqrt{1 + \frac{\gamma}{(l+\frac{1}{2})^2}}.$$

From (5) we get

$$\delta\,(k, l, \gamma) = \frac{\pi}{2}\,(l' - l)$$

and from (9)

$$\frac{\pi}{c\hbar}\,\varDelta E = -\lim_{K=\infty} \sum_{l=0}^{\infty} (2\,l+1)\,\frac{\pi}{2}\,(l'-l) \int_0^K \frac{k_0}{\sqrt{k_0^2 + \varkappa^2}}\,d\,k_0 =$$

$$= -\lim_{K=\infty} \left\{ \sum_{l=0}^{L} (2\,l+1)\,\frac{\pi}{2}\,(l'-l) \int_0^K \frac{k_0}{\sqrt{k_0^2+\varkappa^2}}\,d\,k_0 + \right.$$

$$\left. + \sum_{l=L}^{\infty} (2\,l+1)\,\frac{\pi}{2} \left(\frac{1}{2}\,\frac{\gamma}{l+\frac{1}{2}} - \frac{1}{8}\,\frac{\gamma^2}{(l+\frac{1}{2})^3} + \cdots \right) \int_0^K \frac{k_0}{\sqrt{k_0^2+\varkappa^2}}\,d\,k_0 \right\} =$$

$$= -\lim_{K=\infty} \left\{ \sum_{l=0}^{L} (2\,l+1)\,\frac{\pi}{2}\,(l'-l) \int_0^K \frac{k_0}{\sqrt{k_0^2+\varkappa^2}}\,d\,k_0 + \right.$$

$$\left. + \frac{\pi}{2} \sum_{l=L}^{\infty} \gamma \int_0^K \frac{k_0}{\sqrt{k_0^2+\varkappa^2}}\,d\,k_0 - \frac{\pi}{2}\,\gamma^2 \sum_{l=L}^{\infty} \frac{1}{(2\,l+1)^2} \int_0^K \frac{k_0}{\sqrt{k_0^2+\varkappa^2}}\,d\,k_0 + \cdots \right\}.$$

Thus a finite K would give an infinite self-energy but a finite interaction energy. When K tends to infinity the interaction energy tends linearly to infinity.

(C). We put

$$\left.\begin{array}{ll} \Omega = 1 & r \leq a \\ \Omega = 0 & r > a \end{array}\right\}$$

and get the following differential equation

$$\left\{\frac{d^2}{d r^2} + \left(k_\gamma^2 - \frac{l(l+1)}{r^2}\right)\right\} U(r) = 0 \quad \text{valid for } r \leq a$$
$$\left\{\frac{d^2}{d r^2} + \left(k^2 - \frac{l(l+1)}{r^2}\right)\right\} U(r) = 0 \quad \text{valid for } r > a \tag{2 C}$$

where

$$k_\gamma = \sqrt{k^2 - \gamma}\,.$$

These equations too are transformed BESSEL's[1] equations and their solutions are $\sqrt{r}$ times BESSEL functions of the order of $l + \frac{1}{2}$. As U must be $= 0$ at the origin, the inner solution (where $r \leq a$) is

$$U_i(r) \sim \sqrt{k_\gamma r} \cdot J_{l+\frac{1}{2}}(k_\gamma r).$$

The outer solution ($r > a$) is

$$U_e(r) \sim \sqrt{k r}\,[J_{l+\frac{1}{2}}(k r) - c\, Y_{l+\frac{1}{2}}(k r)]$$

where

$$Y_{l+\frac{1}{2}}(k r) = (-1)^{l+1} J_{-l-\frac{1}{2}}(k r)$$

is WEBER's function and c is a constant that we shall determine. The continuity of the solution and its first derivative requires:

$$\left[\frac{d\,U_i(r)}{d r}\Big/ U_i(r)\right]_{r=a} = \left[\frac{d\,U_e(r)}{d r}\Big/ U_e(r)\right]_{r=a}$$

where

$$\frac{d\,U_i(r)}{d r} = \frac{k_\gamma}{\sqrt{k_\gamma r}}\left(-l\,J_{l+\frac{1}{2}}(k_\gamma r) + k_\gamma r\, J_{l-\frac{1}{2}}(k_\gamma r)\right)$$

$$\frac{d\,U_e(r)}{d r} = \frac{k}{\sqrt{k r}}\left[-l\left(J_{l+\frac{1}{2}}(k r) - c\, Y_{l+\frac{1}{2}}(k r)\right) + k r\left(J_{l-\frac{1}{2}}(k r) - c\, Y_{l-\frac{1}{2}}(k r)\right)\right].$$

From these equations we can derive c and from (5), (3) and

$$\sqrt{kR}\, Y_{l+\frac{1}{2}}(k R) \approx -\sqrt{\frac{2}{\pi}} \cos\left(k R - l \frac{\pi}{2}\right)$$

we get

$$\delta(k, l, \gamma) = \operatorname{arctg} c.$$

We thus get the energy (9)

$$\frac{\pi}{c\hbar}\, \Delta E = -\sum_{l=0}^{\infty} (2l+1) \int_0^\infty \frac{k_0\, d k_0}{\sqrt{k_0^2 + \varkappa^2}} \cdot$$
$$\cdot \operatorname{arctg} \frac{k_0 a\, J_{l-\frac{1}{2}}(k_0 a)\, J_{l+\frac{1}{2}}(k_{0\gamma} a) - k_{0\gamma} a\, J_{l-\frac{1}{2}}(k_{0\gamma} a)\, J_{l+\frac{1}{2}}(k_0 a)}{k_0 a\, Y_{l-\frac{1}{2}}(k_0 a)\, J_{l+\frac{1}{2}}(k_{0\gamma} a) - k_{0\gamma} a\, J_{l-\frac{1}{2}}(k_{0\gamma} a)\, Y_{l+\frac{1}{2}}(k_0 a)} \tag{9 C}$$

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[1] WATSON, *Theory of* BESSEL *Functions*, 1:st ed., 1922.

with

$$k_{0\gamma} = \sqrt{k_0^2 - \gamma}.$$

In order to investigate the convergence of this expression we can divide it into four kinds of integrals

$$(\alpha) = \sum_{l=0}^{L} \int_0^K, \quad (\beta) = \sum_{L}^{\infty} \int_0^K, \quad (\gamma) = \sum_{0}^{L} \int_0^{\infty} \text{ and } (\delta) = \sum_{L}^{\infty} \int_K^{\infty}.$$

(α) is convergent as it has finite limits and its integrand is continuous (at least if γ is not too large).

(β) is convergent. In fact[1]

$$J_\nu(k_0 a) \sim \frac{e^{\nu(\operatorname{tgh}\alpha - \alpha)}}{\sqrt{2\pi\nu \operatorname{tgh}\alpha}} \left(1 + 0\left(\frac{1}{\nu}\right)\right)$$

$$Y_\nu(k_0 a) \sim \frac{e^{-\nu(\operatorname{tgh}\alpha - \alpha)}}{\sqrt{\frac{1}{2}\pi\nu \operatorname{tgh}\alpha}} \left(1 + 0\left(\frac{1}{\nu}\right)\right)$$

where $\alpha = \operatorname{arccosh} \dfrac{\nu}{a k_0}$ and similarly for

$$J_\nu(k_{0\gamma} a).$$

These asymptotic expansions are valid for large ν and $a k_0 = \nu(1 - \varepsilon)$ where ε is any number > 0.

Introducing these expansions in (9 C) the integrand will have as factor

$$e^{-2l(\alpha - \operatorname{tgh}\alpha)}.$$

The terms in the infinite series of (β) thus tend exponentially towards 0 and we get a convergent series. We may even let K tend to infinity.

If k_0 and $k_{0\gamma}$ are large we can use LOMMEL's expansions of the BESSEL functions[2]

$$(a k_{0\gamma})^{-\nu} J_\nu(a k_{0\gamma}) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2} a^2 \gamma)^m}{m!} (a k_0)^{-(\nu+m)} J_{\nu+m}(a k_0).$$

Expanding too the arctg c and $\sqrt{k_0^2 + \varkappa^2}$ we get for (γ) and (δ) the integrand of $\Delta E =$

$$\begin{aligned}
&= \left(1 - \tfrac{1}{2}\frac{\varkappa^2}{k_0^2} + \cdots\right) \cdot \Big\{\gamma a^2 \left(J_{l-\frac{1}{2}}(a k_0) J_{l+\frac{3}{2}}(a k_0) - J_{l+\frac{1}{2}}(a k_0)^2\right) + \\
&+ (\gamma a^2)^2 \Big[\frac{1}{4 a k_0} \left(J_{l-\frac{1}{2}}(a k_0) J_{l+\frac{5}{2}}(a k_0) - J_{l+\frac{1}{2}}(a k_0) J_{l+\frac{3}{2}}(a k_0)\right) - \\
&- \frac{\pi}{4} \left(J_{l-\frac{1}{2}}(a k_0) J_{l+\frac{3}{2}}(a k_0) - J_{l+\frac{1}{2}}(a k_0)^2\right) \left(Y_{l-\frac{1}{2}}(a k_0) J_{l+\frac{3}{2}}(a k_0) - \right. \\
&\left. - Y_{l+\frac{1}{2}}(a k_0) J_{l+\frac{1}{2}}(a k_0)\right) \Big] + \cdots \Big\}.
\end{aligned}$$

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[1] WATSON, loc. cit. § 8: 4.

[2] loc. cit. § 5: 22.

Assuming in (γ) that $k_0 \gg l$ we can introduce here the expressions of $J_{l+\frac{1}{2}}$ and $Y_{l+\frac{1}{2}}$ in a finite series and stop at the second term.

$$J_{l+\frac{1}{2}}(a k_0) = + \sqrt{\frac{2}{\pi a k_0}} \sin\left(a k_0 - \frac{l\pi}{2}\right) + \sqrt{\frac{2}{\pi a k_0}} \cdot \frac{l^2 + l}{2 a k_0} \cos\left(a k_0 - \frac{l\pi}{2}\right) + \cdots$$

$$Y_{l+\frac{1}{2}}(a k_0) = - \sqrt{\frac{2}{\pi a k_0}} \cos\left(a k_0 - \frac{l\pi}{2}\right) + \sqrt{\frac{2}{\pi a k_0}} \cdot \frac{l^2 + l}{2 a k_0} \sin\left(a k_0 - \frac{l\pi}{2}\right) + \cdots$$

and we get the integrand =

$$-\frac{\pi}{2}\left\{\gamma a^2 \cdot \frac{2}{\pi a k_0}\left(-1 + 0\left(\frac{l}{k_0}\right)\right) - (\gamma a^2)^2 \frac{1}{\pi (a k_0)^3}\left(\frac{1}{2} - 2\cos^2\left(a k_0 - \frac{l\pi}{2}\right)\right)\right\}.$$

We have thus seen that the terms of ΔE linear in γ have a logarithmic divergence, while the other terms converge. Thus from (γ) we get an infinite self-energy but the contribution to the interaction energy is finite. This is true even if L tends to infinity.

In (δ) we cannot stop the expansion of the Bessel functions in the same way as in (γ). It is, however, possible to integrate one of the γ^2-terms of ΔE We have[1]

$$\lim_{K'=\infty} \int_K^{K'} J_{\mu+\frac{1}{2}}(a k) J_{\nu+\frac{1}{2}}(a k) \cdot \frac{dk}{k} =$$

$$= \lim_{K'=\infty} \int_K^{K'} \left(-\frac{a k \{J_{\mu+\frac{3}{2}}(a k) J_{\nu+\frac{1}{2}}(a k) - J_{\mu+\frac{1}{2}}(a k) J_{\nu+\frac{3}{2}}(a k)\}}{(\mu - \nu)(\mu + \nu + 1)} + \right.$$

$$\left. + \frac{J_{\mu+\frac{1}{2}}(a k) J_{\nu+\frac{1}{2}}(a k)}{\mu + \nu + 1}\right) = -\frac{2}{\pi} \frac{\sin(\nu - \mu)\frac{\pi}{2}}{(\mu - \nu)(\mu + \nu + 1)} +$$

$$+ \frac{a K \{J_{\mu+\frac{3}{2}}(a K) J_{\nu+\frac{1}{2}}(a K) - J_{\mu+\frac{1}{2}}(a K) J_{\nu+\frac{3}{2}}(a K)\}}{(\mu - \nu)(\mu + \nu + 1)} - \frac{J_{\mu+\frac{1}{2}}(a K) J_{\nu+\frac{1}{2}}(a K)}{\mu + \nu + 1};$$

where we have introduced the expansion of $J_{l+\frac{1}{2}}(a k)$ given above. Thus

$$\sum_L^{\infty} (2l + 1) \int_K^{\infty} \frac{d k_0}{k_0} \left(J_{l-\frac{1}{2}}(a k_0) J_{l+\frac{5}{2}}(a k_0) - J_{l+\frac{1}{2}}(a k_0) J_{l+\frac{3}{2}}(a k_0)\right) =$$

$$= -\frac{2}{\pi} \sum_L^{\infty} (2l + 1) \left\{\frac{1}{3 \cdot 2 (l + 1)} + \frac{1}{1 \cdot 2 (l + 1)} + \right.$$

$$\left. + \text{ terms containing Bessel functions}\right\}.$$

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[1] Watson, loc. cit. p. 135.

As we saw in (β) the terms containing BESSEL functions tend exponentially towards 0. We get the series

$$\sum_{L}^{\infty} \frac{2\,l+1}{l+1} \text{ which is not convergent.}$$

We see that in (δ) at least one of the γ^2-terms diverges. Thus even the interaction energy has at least a linear divergency, arising from the region where $a\,k$ and l are roughly of the same order of magnitude and both tend to infinity.

We see that the assumption of a source of finite extension in ordinary space does not give either a finite self-energy or a finite interaction energy — at least not if the sources coincide. Thus we have to make a more radical assumption than this one to get rid of the divergency difficulties in the pair theory even if we are not interested in the relativistic invariance of the assumption or in the convergence of the self-energy problem. We must assume that for some reason $a\,k$ and l are not allowed to tend to infinity, at least not at the same time and in the same way, i. e. we must introduce some kind of "cutting-off".

§ 2. A criterion for the validity of the pertubation theory

As above we shall assume fixed, extended sources and in addition some kind of "cutting-off" rule that makes the total interaction energy converge. We shall, however, see that if we restrict ourselves to a perturbation of the second order, this rule need not be stated very definitely. We can get a criterion for the validity of the perturbation theory if, for a fixed value of the unperturbed energy, we calculate the corresponding value of the interaction energy using the perturbation of the second order, and compare this value with that obtained by including the perturbation of the third order. A criterion for scalar mesons and scalar interaction has been derived by WENTZEL [42] and the corresponding criterion assuming another spin-independent interaction has been derived by PAULI and HU [45]. The latter authors have derived a criterion for vector mesons too, assuming a spin-independent interaction. The kind of extended source they assume is, however, not the same as ours (see note 1 p. 147). KLEIN[1] has derived the same criterion as WENTZEL using the same kind of extended source as we. In this paper we shall use a modification of his method, limiting ourselves to one spin-dependent interaction in the case of scalar mesons and one in the case of vector mesons. The criterions for the other interactions can be derived in the same way. Starting from the generalized DIRAC equations for the meson and eliminating the components of the field function, with the exception of a scalar in the scalar case and a vector in the vector case, we can get the KLEIN-GORDON equation and the PROCA equation (with three components only) respectively. The right members of the equations are of course not zero any longer, but some function of space, homogeneous in the field function. Starting from a set of solutions of the unperturbed KLEIN-

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[1] Not published. I want to thank Professor KLEIN for his kindness of giving me the opportunity of seeing this paper.

GORDON and PROCA equations, respectively, we can, with the aid of a GREEN function, write the solution of the differential equation as a power series of the interaction constant. By this method we can obtain the values of the interaction energy as a function of the unperturbed energy in any approximation. Half the sum of the moduli of these energy values give the total interaction energy (cf. Ch. II § 1 and HJALMARS [49]). As the unperturbed energy is degenerated we ought to have used a set of solutions, particularly chosen for the perturbation terms. We shall see, however, that this sum is independent of our particular choice of solutions and we have, therefore, a complete freedom of choosing any set of unperturbed solutions.

We put the wave function of the meson field $= U(\boldsymbol{r}, t)$, being a scalar, if we have scalar mesons, and a vector, if we have vector mesons. The source function is described by $\gamma\,\Omega$, where γ is the interaction constant. Ω is of course a function of space but it may be some kind of operator too, operating upon U, as e. g. grad or $\frac{1}{c}\frac{\partial}{\partial t}$ or mixing the different components of U. It may even form a series in ascending powers of γ.

The field equation is

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \varkappa^2 U = \gamma\,\Omega\,U \tag{12}$$

which is a scalar equation (one equation) in the case of scalar mesons and a vector equation (three simultaneous equations corresponding to the three different components of U) in the case of vector mesons. In this case, usually the three components are mixed in the right members of the equation.

We imagine the system included in a cube whose side is very large $= A$ and whose volume $A^3 = V$. Then we get the values of the energy states of the system (E) if we substitute

$$U(\boldsymbol{r}, t) = U(\boldsymbol{r}) \cdot e^{-\frac{i}{\hbar}Et} \tag{13}$$

in (12).

Thus the field equation is

$$\left(\Delta + \left(\frac{E}{c\hbar}\right)^2 - \varkappa^2\right) U(\boldsymbol{r}) = \gamma\,\Omega(\boldsymbol{r})\,U(\boldsymbol{r}) \tag{14}$$

where Ω may also be a function of E.

A solution of the unperturbed field equation is

$$U_{0k\nu}(\boldsymbol{r}, t) = U_{0k\nu}(\boldsymbol{r})\,e^{-\frac{iE_0}{\hbar}t} = U_{0k\nu}\,e^{i\left(kr - \frac{E_0}{\hbar}t\right)} \tag{15}$$

where $\nu = 1, 2, 3$ corresponds to the three different fundamental solutions, corresponding to a single value of $\boldsymbol{k}$ in the vector case. If

$$\int_V U^*_{0k\mu}(\boldsymbol{r})\,U_{0k\nu}(\boldsymbol{r})\,dV = \delta_{\mu\nu} \tag{15 A}$$

we have an orthogonal and normalized fundamental system of solutions. Putting (15) in (12) with $\gamma = 0$ we get

$$E_0 = c\hbar\sqrt{|\boldsymbol{k}|^2 + \varkappa^2} \tag{16}$$

and thus E_0 is a function of $|\boldsymbol{k}|$ only and it is degenerated with respect to the direction of $\boldsymbol{k}$ and in the vector case with respect to ν too. For every value of $|\boldsymbol{k}|$ there are two symmetrical values of E_0. As above, we have dropped the negative value of E_0. $U_{0k\nu}$ is a constant scalar or vector which we assume to be suitably chosen as regards the perturbation. $\boldsymbol{k}$ is determined by the fact that $U_{0k\nu}$ shall be a periodic function of the side of the cube (this is an unessential modification of our previous condition, that the wave function must be zero on the surface of the space, permitting us to use exponential functions instead of trigonometrical ones). The real solution $U_{k\nu}(\boldsymbol{r})$ must of course also be a periodic function of the cube. As we assume that the sources are concentrated in a relatively small region around the centre of the cube, the difference between $U_{k\nu}(\boldsymbol{r})$ and $U_{0k\nu}(\boldsymbol{r})$ must be small, if we are far away from the centre.

We multiply (14) from the left by $U^*_{0k\nu}(\boldsymbol{r})$ and integrate over the volume of the cube. By partial integrations, bearing in mind that the surface integrals are $= 0$ on account of the periodicity of $U_{k\nu}(\boldsymbol{r})$ and $U_{0k\nu}(\boldsymbol{r})$, we get, if we omit for the moment the indices $\boldsymbol{k}$ and ν,

$$\int\limits_V U_0^*(\boldsymbol{r})\,(\varDelta\, U(\boldsymbol{r}))\,dV = \int\limits_V (U_0^*(\boldsymbol{r})\,\varDelta)\,U(\boldsymbol{r})\,dV. \tag{17}$$

At the same time we shall expand E, $U(\boldsymbol{r})$ in ascending powers of γ and write down the possible expansion of $\Omega(\boldsymbol{r})$ i. e.

$$E = E_0 + \gamma E_1 + \gamma^2 E_2 + \gamma^3 E_3 + \cdots \tag{18}$$

$$U(\boldsymbol{r}) = U_0(\boldsymbol{r}) + \gamma U_1(\boldsymbol{r}) + \gamma^2 U_2(\boldsymbol{r}) + \cdots \tag{19}$$

and

$$\Omega(\boldsymbol{r}) = \Omega_1(\boldsymbol{r}) + \gamma\,\Omega_2(\boldsymbol{r}) + \gamma^2\,\Omega_3(\boldsymbol{r}) + \cdots. \tag{20}$$

Thus we get

$$\begin{aligned}\int\limits_V \Bigg\{ & \left[U_0^*(\boldsymbol{r})\left(\varDelta + \left(\frac{E_0}{c\hbar}\right)^2 + \varkappa^2\right)\right] U(\boldsymbol{r}) + \\ & + \gamma\, U_0^*(\boldsymbol{r})\left[2\,\frac{E_0 E_1}{(c\hbar)^2}\,U_0(\boldsymbol{r}) - \Omega_1(\boldsymbol{r})\,U_0(\boldsymbol{r})\right] + \\ & + \gamma^2\, U_0^*(\boldsymbol{r})\left[\frac{1}{(c\hbar)^2}\,(2E_0E_2 + E_1^2)\,U_0(\boldsymbol{r}) + 2\,\frac{E_0E_1}{(\hbar c)^2}\,U_1(\boldsymbol{r}) - \Omega_2(\boldsymbol{r})\,U_0(\boldsymbol{r}) - \right. \\ & \qquad\qquad \left. - \Omega_1(\boldsymbol{r})\,U_1(\boldsymbol{r})\right] + \\ & + \gamma^3\, U_0^*(\boldsymbol{r})\left[\frac{2}{(\hbar c)^2}\,(E_0E_3 + E_1E_2)\,U_0(\boldsymbol{r}) + \frac{1}{(\hbar c)^2}\,(2E_0E_2 + E_1^2)\,U_1(\boldsymbol{r}) + \right. \\ & \left. + 2\,\frac{E_0E_1}{(\hbar c)^2}\,U_2(\boldsymbol{r}) - \Omega_3(\boldsymbol{r})\,U_0(\boldsymbol{r}) - \Omega_2(\boldsymbol{r})\,U_1(\boldsymbol{r}) - \Omega_1(\boldsymbol{r})\,U_2(\boldsymbol{r})\right] + \cdots \Bigg\}\,dV = 0. \end{aligned} \tag{21}$$

(21) shall be valid for every value of γ i. e. the integral of every bracket in

it must be $=0$. As $U_0^*(\boldsymbol{r})$ is a solution of the unperturbed equation the first bracket is identically $=0$. Remembering (15 A) and (16) we get the energy

$$E_1 = \frac{c\hbar}{2} \frac{1}{\sqrt{k^2+\varkappa^2}} \int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_0(\boldsymbol{r}))\,d\,V$$

$$\begin{aligned} E_2 = &-\frac{E_1^2}{2\,E_0} - E_1 \int_V U_0^*(\boldsymbol{r})\,U_1(\boldsymbol{r})\,d\,V + \\ &+\frac{c\hbar}{2}\frac{1}{\sqrt{k^2+\varkappa^2}}\left[\int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_1(\boldsymbol{r}))\,d\,V + \int_V U_0^*(\boldsymbol{r})\,(\Omega_2(\boldsymbol{r})\,U_0(\boldsymbol{r}))\,d\,V\right] \end{aligned} \tag{22}$$

$$\begin{aligned} E_3 = &-\frac{E_1\,E_2}{E_0} - \frac{2\,E_0\,E_2+E_1^2}{2\,E_0}\int_V U_0^*(\boldsymbol{r})\,U_1(\boldsymbol{r})\,d\,V - E_1\int_V U_0^*(\boldsymbol{r})\,U_2(\boldsymbol{r})\,d\,V + \\ &+\frac{c\hbar}{2}\frac{1}{\sqrt{k^2+\varkappa^2}}\left[\int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_2(\boldsymbol{r}))\,d\,V + \int_V U_0^*(\boldsymbol{r})\,(\Omega_2(\boldsymbol{r})\,U_1(\boldsymbol{r}))\,d\,V + \right. \\ &\left. + \int_V U_0^*(\boldsymbol{r})\,(\Omega_3(\boldsymbol{r})\,U_0(\boldsymbol{r}))\,d\,V\right]. \end{aligned}$$

As we are only interested in the interaction energy of the sources we may neglect the self-energy terms, i. e. all terms which do not contain the distances between the sources such as $\int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_0(\boldsymbol{r}))\,d\,V$. As $U_1(\boldsymbol{r})$, $U_2(\boldsymbol{r})$ etc. are mainly of the same order of magnitude as $U_0(\boldsymbol{r})$ inside the region of the sources and small outside it, some of the integrals containing these functions may easily be seen to be negligibly small.

Neglecting in the energy all terms smaller than $\frac{1}{V}$, we get from (22) the interaction energy of the first, second and third order of perturbation

$$\Delta\,E_1 = 0$$

$$\Delta\,E_2 = \frac{c\hbar}{2}\frac{1}{\sqrt{k^2+\varkappa^2}}\int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_1(\boldsymbol{r}))\,d\,V \tag{23}$$

$$\Delta\,E_3 = \frac{c\hbar}{2}\frac{1}{\sqrt{k^2+\varkappa^2}}\left[\int_V U_0^*(\boldsymbol{r})\,(\Omega_1(\boldsymbol{r})\,U_2(\boldsymbol{r}))\,dV + \int_V U_0^*(\boldsymbol{r})\,(\Omega_2(\boldsymbol{r})\,U_1(\boldsymbol{r}))\,dV\right] \tag{24}$$

with

$$\Delta\,E = \gamma\,\Delta\,E_1 + \gamma^2\,\Delta\,E_2 + \gamma^3\,\Delta\,E_3 + \cdots.$$

The $\Delta\,E$:s have, of course, generally the indices $\boldsymbol{k}$, ν too. We obtain the total interaction energy in the approximation of a perturbation of the second and third order, if we take the sums of $\Delta\,E_{2k\nu}$ and $\Delta\,E_{3k\nu}$, summing for every value of $\boldsymbol{k}$ and ν, remembering our cutting-off rule, and letting the side of the cube tend to infinity. In (23) and (24) some self-energies are still left, which we shall neglect.

In order to determine $U(\boldsymbol{r})$ we put the expansions (18), (19), and (20) into (14)

$$\left[\Delta + \left(\frac{1}{c\hbar}\right)^2 \{E_0^2 + \gamma 2 E_0 E_1 + \gamma^2 (E_1^2 + 2 E_0 E_2) + \cdots\} - \varkappa^2 - \gamma\, \Omega_1(\boldsymbol{r}) - \gamma^2\, \Omega_2(\boldsymbol{r}) - \cdots\right] \cdot$$
$$\cdot (U_0(\boldsymbol{r}) + \gamma\, U_1(\boldsymbol{r}) + \gamma^2\, U_2(\boldsymbol{r}) + \cdots) = 0 \qquad (25)$$

or

$$\left(\Delta + \left(\frac{E_0^2}{c\hbar}\right)^2 - \varkappa^2\right) U_0(\boldsymbol{r}) = 0$$

which is satisfied by (15) and (16) and

$$\left(\Delta + \left(\frac{E_0}{c\hbar}\right)^2 - \varkappa^2\right) U_1(\boldsymbol{r}) = \left(\Omega_1(\boldsymbol{r}) - \frac{2 E_0 E_1}{(c\hbar)^2}\right) U_0(\boldsymbol{r})$$

$$\left(\Delta + \left(\frac{E_0}{c\hbar}\right)^2 - \varkappa^2\right) U_2(\boldsymbol{r}) = \left(\Omega_1(\boldsymbol{r}) - \frac{2 E_0 E_1}{(c\hbar)^2}\right) U_1(\boldsymbol{r}) +$$
$$+ \left(\Omega_2(\boldsymbol{r}) - \frac{E_1^2 + 2 E_0 E_2}{(c\hbar)^2}\right) U_0(\boldsymbol{r}).$$

Neglecting in the right members the terms that disappear when V tends to infinity we get

$$\begin{aligned} (\Delta + k^2)\, U_1(\boldsymbol{r}) &= \Omega_1(\boldsymbol{r})\, U_0(\boldsymbol{r}) \\ (\Delta + k^2)\, U_2(\boldsymbol{r}) &= \Omega_1(\boldsymbol{r})\, U_1(\boldsymbol{r}) + \Omega_2(\boldsymbol{r})\, U_0(\boldsymbol{r}). \end{aligned} \qquad (26)$$

A solution of the equation

$$(\Delta + k^2)\, U(\boldsymbol{r}) = f(\boldsymbol{r})$$

where the function $f(\boldsymbol{r})$ is known, is

$$U(\boldsymbol{r}) = \int_V G_V(\boldsymbol{r}, \boldsymbol{r}')\, f(\boldsymbol{r}')\, dV'.$$

If we put the GREEN function

$$G_V(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{V} \sum_{k'} \frac{e^{ik'(r-r')}}{k^2 - k'^2} \qquad (27)$$

the condition that $U(\boldsymbol{r})$ shall be periodic in the cube is fulfilled. Remembering that

$$\frac{1}{V} \int_V \sum_{k'} e^{ik'(r-r')} f(\boldsymbol{r}')\, dV' = f(\boldsymbol{r})$$

we can easily see that we have got a solution of the differential equation.

Letting the cube tend to infinity, we have

$$G(\boldsymbol{r}, \boldsymbol{r}') = \lim_{V=\infty} G_V(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{(2\pi)^3} \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} \frac{e^{ik'|r-r'|u}}{k^2 - k'^2}\, k'^2\, dk'\, du\, d\varphi$$

where $\boldsymbol{r} - \boldsymbol{r}'$ is the polar axis and $u = \cos\vartheta$.

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{2\,\pi^2}\,\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\int\limits_0^\infty \frac{k' \sin k' |\boldsymbol{r} - \boldsymbol{r}'|}{k^2 - k'^2}\, d\,k'.$$

Taking the principal value of this integral, we get

$$G(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{4\,\pi}\,\frac{\cos k\,|\boldsymbol{r} - \boldsymbol{r}'|}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{28}$$

We see that if $f(\boldsymbol{r}) \neq 0$ only within a limited region $\lim\limits_{r\ \ \infty} U(r) = 0$ as we have required above. Thus we get from (26)

$$U_1(\boldsymbol{r}) = \int\limits_V G_V(\boldsymbol{r}, \boldsymbol{r}')\,(\Omega_1(\boldsymbol{r}')\,U_0(\boldsymbol{r}'))\,d\,V' \tag{29}$$

and

$$\begin{aligned} U_2(\boldsymbol{r}) &= \int\limits_V G_V(\boldsymbol{r}, \boldsymbol{r}')\,(\Omega_1(\boldsymbol{r}')\,U_1(\boldsymbol{r}')\,d\,V' + \\ &+ \int\limits_V G_V(\boldsymbol{r}, \boldsymbol{r}')\,(\Omega_2(\boldsymbol{r}')\,U_0(\boldsymbol{r}'))\,d\,V' = \\ &= \int\limits_V \int\limits_V G_V(\boldsymbol{r}, \boldsymbol{r}')\,[\Omega_1(\boldsymbol{r}')\,G_V(\boldsymbol{r}', \boldsymbol{r}'')\,(\Omega_1(\boldsymbol{r}'')\,U_0(\boldsymbol{r}''))]\,d\,V'\,d\,V'' + \\ &+ \int\limits_V G_V(\boldsymbol{r}, \boldsymbol{r}')\,(\Omega_2(\boldsymbol{r}')\,U_0(\boldsymbol{r}'))\,d\,V'. \end{aligned} \tag{30}$$

The criterion mentioned above is obtained from the requirement that

$$\Delta\,E_{2\,k\,\nu} \geqslant \gamma\,\Delta\,E_{3\,k\,\nu} \tag{31}$$

where $\Delta\,E_{2\,k\,\nu}$ is given by (23, 29) and $\Delta\,E_{3\,k\,\nu}$ by (24, 30). As the second term of (24) is analogous to the part of the first term of (24) arising from the second term of $U_2(\boldsymbol{r})$ in (30), we may omit this term when we determine the criterion.

We shall now show that it is permissible to choose any fundamental set of solutions of the unperturbed equations if we want to calculate the total interaction energy. We shall here limit ourselves to the perturbation of the second order, but the proof can easily be generalized to any perturbation of higher order.

$U_{0\lambda}(\boldsymbol{r})$ is an arbitrary ortho-normalized fundamental set of solutions of the unperturbed KLEIN-GORDON or PROCA equation and $U_{0l}(\boldsymbol{r})$ is another ortho-normalized fundamental set of solutions which is particularly chosen for the perturbation calculation. Both sets correspond to the same energy $E_{0\,|\,k\,|}$.

Thus we can put

$$U_{0l}(\boldsymbol{r}) = \sum_{\lambda} c_{l\lambda}\,U_{0\lambda}(\boldsymbol{r}) \tag{32}$$

where $c_{l\lambda}$ are constants.

The ortho-normalization condition gives

$$\int_V U_{0m}^*(\boldsymbol{r})\, U_{0l}(\boldsymbol{r})\, d\,V = \sum_{\mu\lambda} c_{m\mu}^*\, c_{l\lambda} \int_V U_{0\mu}^*(\boldsymbol{r})\, U_{0\lambda}(\boldsymbol{r})\, d\,V =$$

$$= \sum_{\mu\lambda} c_{m\mu}^*\, c_{l\lambda}\, \delta_{\mu\lambda} = \sum_{\lambda} c_{m\lambda}^*\, c_{l\lambda} = \delta_{ml} \qquad (33)$$

and correspondingly

$$\sum_{m} c_{m\mu}^*\, c_{m\lambda} = \delta_{\mu\lambda}. \qquad (34)$$

From (23) and (29) we get for the total interaction energy

$$\varDelta\, E_{2i} = \sum_{lk} \varDelta\, E_{2lk} = \sum_{lk} \frac{\hbar\, c}{2} \frac{1}{\sqrt{k^2 + \varkappa^2}} \cdot$$

$$\cdot \iint U_{0l}^*(\boldsymbol{r}')\, [\Omega_1(\boldsymbol{r}')\, G(\boldsymbol{r}', \boldsymbol{r}'')\, (\Omega_1(\boldsymbol{r}'')\, U_{0l}(\boldsymbol{r}''))]\, d\,V'\, d\,V'' \qquad (35)$$

and from (32)

$$\varDelta\, E_{2i} = \sum_{l,\lambda,\mu,k} c_{l\lambda}^*\, c_{l\mu} \frac{\hbar\, c}{2} \frac{1}{\sqrt{k^2 + \varkappa^2}} \cdot$$

$$\cdot \iint U_{0\lambda}^*(\boldsymbol{r}')\, [\Omega_1(\boldsymbol{r}')\, G(\boldsymbol{r}', \boldsymbol{r}'')\, (\Omega_1(\boldsymbol{r}'')\, U_{0\mu}(\boldsymbol{r}''))]\, d\,V'\, d\,V''.$$

From (34) we get

$$\varDelta\, E_{2i} = \sum_{\lambda,k} \frac{\hbar\, c}{2} \frac{1}{\sqrt{k^2 + \varkappa^2}} \cdot$$

$$\cdot \iint U_{0\lambda}^*(\boldsymbol{r}')\, [\Omega_1(\boldsymbol{r}')\, G(\boldsymbol{r}', \boldsymbol{r}'')\, (\Omega_1(\boldsymbol{r}'')\, U_{0\lambda}(\boldsymbol{r}''))]\, d\,V'\, d\,V''. \qquad (36)$$

Thus we are allowed to sum, assuming an arbitrary fundamental set of solutions of the unperturbed equation, which of course can always be made orthogonal and normalized.

With this method we shall now deduce a criterion for the validity of the perturbation theory in some particular cases.

A. *Scalar Mesons with spin-dependent interaction*

If we assume the interaction term to be built up of antisymmetric tensors, the generalized Dirac equation of scalar mesons is (cf. Hjalmars [49] 110, 192 and 331)

$$\left[\left(\sum_{k=1}^{4} p_k\, \beta_k + i\, \mu\, c - i\, \gamma\, \mu\, c \sum_{\substack{k,\, l=1 \\ k \neq l}}^{4} \tilde{\psi}\, \gamma_k\, \gamma_l\, \psi\, \beta_k\, \beta_l \right) U \right]_m = 0. \qquad (37)$$

$$m = 1,\, 2,\, 3,\, 4,\, 5$$

We shall assume the source to be at rest and thus neglect the smaller terms of the source function, i. e. the sums in the interaction terms may be taken between 1 and 3 instead of between 1 and 4.

We have further

$$\tilde{\psi}\,\gamma_k\,\gamma_l\,\psi = \psi^*\,\gamma_4\,\gamma_k\,\gamma_l\,\psi = -\,i\,S_{3kl} \qquad (k \neq l = 1,\ 2,\ 3)\ \text{(cf. Ch. I (3))}. \tag{38}$$

The element of the matrix β_k is (cf. HJALMARS [49] equ. 80)

$$(m\,|\,\beta_k\,|\,n) = i\,(\delta_{mk}\,\delta_{5n} - \delta_{m5}\,\delta_{kn}). \tag{39}$$

Introducing this in (37) we get

$$m = 1,\ 2 \text{ or } 3 \qquad i\,p_m\,U_5 + i\,\mu\,c\,U_m - \gamma\,\mu\,c \sum_{l=1}^{3} S_{3ml}\,U_l = 0$$

or

$$\boldsymbol{p}\,U + \mu\,c\,\boldsymbol{U}_g = i\,\gamma\,\mu\,c\,(\boldsymbol{S}_3 \times \boldsymbol{U}_g) \tag{40}$$

where U_5 is put $= U$ and the vector $\boldsymbol{U}_g$ has the components U_l ($l = 1$, 2 and 3).

Further we get from (37)

$$p_4\,U + \mu\,c\,U_4 = 0 \tag{41}$$

and

$$\boldsymbol{p} \cdot \boldsymbol{U}_g + p_4\,U_4 - \mu\,c\,U = 0. \tag{42}$$

From (40) we can express $\boldsymbol{U}_g$ in U. As γ is assumed to be small we shall do this as a power series in γ. By iteration we get

$$\boldsymbol{U}_g = -\frac{\boldsymbol{p}}{\mu\,c}\,U + i\,\gamma\,(\boldsymbol{S}_3 \times \boldsymbol{U}_g) =$$

$$= -\frac{1}{\mu\,c}\,[\boldsymbol{p}\,U + i\,\gamma\,(\boldsymbol{S}_3 \times \boldsymbol{p}\,U) - \gamma^2\,\{\boldsymbol{S}_3 \times (\boldsymbol{S}_3 \times \boldsymbol{p}\,U)\} + \cdots]. \tag{43}$$

Introducing the series (43) and the expression for U_4 obtained from (41) into (42) we get

$$\boldsymbol{p}\,[\boldsymbol{p}\,U - i\,\mu\,c\,\gamma\,(\boldsymbol{S}_3 \times \boldsymbol{U}_g)] + p_4^2\,U + \mu^2\,c^2\,U = 0$$

or with $\boldsymbol{p} = -\,i\,\hbar$ grad, $p_4 = -\dfrac{\hbar}{c}\,\dfrac{\partial}{\partial t}$

$$\left(\varDelta\,U - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \varkappa^2\right) U =$$

$$= -\varkappa\,\gamma\ \mathrm{div}\ (\boldsymbol{S}_3 \times \boldsymbol{U}_g) = i\,\gamma\ \mathrm{div}\ (\boldsymbol{S}_3 \times \ \mathrm{grad}\ U) + \gamma^2\ \mathrm{div}\ (\boldsymbol{S}_3 \times (\boldsymbol{S}_3 \times \mathrm{grad}\ U)). \tag{44}$$

We have thus written the field equation in the form of equation (12) with

$$\Omega_1 = i\ \mathrm{div}\ (\boldsymbol{S}_3 \times \mathrm{grad}\ \ )$$

$$\Omega_2 = \mathrm{div}\ (\boldsymbol{S}_3 \times (\boldsymbol{S}_3 \times \mathrm{grad}\ \ )). \tag{45}$$

. . . .

We shall first give the value of the interaction-energy in the approximation of the second order. We have already calculated this value in Chapter I and we shall show how to obtain the expression given there. Our assumptions regarding the source function are the same as in Chapter I. Thus we get ((15), (20), (23), (27), (29), (45))

$$\Delta E_{2i} = \lim_{V=\infty} \frac{c\hbar}{2} \frac{1}{V} \sum_{k',k''} \frac{1}{\sqrt{k'^2 + \varkappa^2}} \int\limits_V \int\limits_V U^*_{0k'} \, e^{-ik'r'} \, i \operatorname{div}' \left(\boldsymbol{S}_3(\boldsymbol{r}') \times \operatorname{grad}' \frac{e^{ik''(r'-r'')}}{k'^2 - k''^2} \right) \cdot$$

$$\cdot \, i \operatorname{div}'' \left(\boldsymbol{S}_3(\boldsymbol{r}'') \times \operatorname{grad}'' U_{0k'} \, e^{ik'r''} \right) dV' \, dV''$$

where grad′ etc. means derivation with respect to x' etc.

By partial integration we get

$$\Delta E_{2i} = \lim_{V=\infty} \frac{-c\hbar}{2} \frac{1}{V} \sum_{k',k''} \frac{1}{\sqrt{k'^2 + \varkappa^2}} U^*_{0k'} U_{0k'} \frac{1}{k'^2 - k''^2} \cdot$$

$$\cdot \int\limits_V \operatorname{grad} e^{-ik'r'} \left[\boldsymbol{S}_3(\boldsymbol{r}') \times \operatorname{grad} e^{ik''r'} \right] dV' \cdot$$

$$\cdot \int\limits_V \operatorname{grad} e^{-ik''r''} \left[\boldsymbol{S}_3(\boldsymbol{r}'') \times \operatorname{grad} e^{ik'r''} \right] dV''. \tag{46}$$

If $U_{0k}(\boldsymbol{r})$ is ortho-normalized, we may — as we have seen — chose U_{0k} quite arbitrarily. We shall put

$$U_{0k} = \frac{1}{\sqrt{V}}. \tag{47}$$

We see that, except for the factor $\frac{1}{\sqrt{k'^2 + \varkappa^2}}$, every term in (46) changes its sign if we exchange $\boldsymbol{k}'$ and $\boldsymbol{k}''$ and at the same time $\boldsymbol{r}'$ and $\boldsymbol{r}''$. Thus we can write every term in (46) as a difference between two terms both half as large as the original one, having performed this exchange in the last term. If we extend the first term by $\sqrt{k''^2 + \varkappa^2}$ and the last term by $\sqrt{k'^2 + \varkappa^2}$, we can perform the subtraction and abbreviate with $\sqrt{k''^2 + \varkappa^2} - \sqrt{k'^2 + \varkappa^2}$.

Carrying out the differentiations in (46) and rearranging some of the terms in the scalar triple products we get

$$\Delta E_{2i} = \tfrac{1}{2} \frac{c\hbar}{2} \frac{1}{(2\pi)^6} \int dV' \, dV'' \, d\boldsymbol{k}' \, d\boldsymbol{k}'' \cdot$$

$$\cdot \, e^{i(\boldsymbol{k}'-\boldsymbol{k}'')(\boldsymbol{r}''-\boldsymbol{r}')} \frac{1}{\sqrt{k'^2 + \varkappa^2} \sqrt{k''^2 + \varkappa^2} \left(\sqrt{k'^2 + \varkappa^2} + \sqrt{k''^2 + \varkappa^2} \right)} \cdot$$

$$\cdot \left\{ \boldsymbol{S}_3(\boldsymbol{r}') \cdot (\boldsymbol{k}' \times \boldsymbol{k}'') \right\} \left\{ \boldsymbol{S}_3(\boldsymbol{r}'') \cdot (\boldsymbol{k}'' \times \boldsymbol{k}') \right\}. \tag{48}$$

If we put

$$\gamma = -8\pi \frac{e^2}{\hbar c} \frac{1}{\varkappa^3} C'_{s3}. \tag{49}$$

$\gamma^2 \Delta E_{2i}$ is easily seen to be identical with $S_{3'3'}$, in Chapter I. We can for example carry out the integrations with respect to φ' and φ''. Then we get (68 $S_{3'3'}$) of Chapter I.

We saw in Chapter I, which cutting-off rule we had to introduce in order to get convergent integrals in (48). If, however, the source functions defined later on in this chapter cut each others, this rule is not sufficient any longer as the integrals obtained do not converge in this case. In any case, if the distance between the sources is very small, the whole theory can not be assumed to be very satisfactory. Thus in our numerical applications we shall have to make an a posteriori assumption concerning the interaction energy inside some small distance, the "cutting-off distance", which is of course of the same order of magnitude as the "radius" of the source defined later on in this chapter.

We shall now investigate the criterion for the validity of the perturbation theory. Owing to the cutting-off for large k, we only need to determine the energy for a fixed value of k which we may assume to be rather small. At present, however, we have to make some explicit assumptions regarding the source. We shall assume that the spin of the source is constant and that we have a spherically symmetrical source function ("radius" $= a$). Thus we put

$$S_3(\boldsymbol{r}) = \sum_{\mu} \sigma_\mu f(|\boldsymbol{r} - \boldsymbol{r}_\mu|), \quad \int_V f(|\boldsymbol{r} - \boldsymbol{r}_\mu|)\, dV = 1 \tag{50}$$

where the constant vector $\boldsymbol{r}_\mu$ determines the position of the μ:th source and $f_\mu(|\boldsymbol{r} - \boldsymbol{r}_\mu|) \equiv 0$ when $|\boldsymbol{r} - \boldsymbol{r}_\mu| > a$.

For the sake of simplicity we shall put $f_\mu(|\boldsymbol{r} - \boldsymbol{r}_\mu|) = \delta(|\boldsymbol{r} - \boldsymbol{r}_\mu|)$ so far as we get a convergent result in that way. In (23) and (24) we may thus by (45) introduce a δ-function and evaluate the integrals. From (31) it follows that the criterion is given by

$$U_{1k}(\boldsymbol{r}_\mu) \gg \gamma\, U_{2k}(\boldsymbol{r}_\mu) \tag{51}$$

where we may particularly put $\boldsymbol{r}_\mu = \boldsymbol{r}_1$. In our calculations we shall assume the distance between the sources to be $\gg a$. We shall only consider the case of two sources S_1 and S_2.

From (28), (29), (30) and (45), we get (omitting the index $\boldsymbol{k}$ in $\boldsymbol{U}_k(\boldsymbol{r})$)

$$U_1(\boldsymbol{r}_1) = -\frac{i}{4\pi}\int \frac{\cos k|\boldsymbol{r}_1 - \boldsymbol{r}'|}{|\boldsymbol{r}_1 - \boldsymbol{r}'|}\, \mathrm{div}'\, (\boldsymbol{S}_3(\boldsymbol{r}') \times \mathrm{grad}'\, U_0(\boldsymbol{r}'))\, dV'.$$

If we omit the terms that do not contain $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ we get with the aid of (15) and (50)

$$\Delta U_1(\boldsymbol{r}_1) = -\frac{1}{4\pi} \operatorname*{grad}_{r=r_2} \frac{\cos k|\boldsymbol{r}_1 - \boldsymbol{r}|}{|\boldsymbol{r}_1 - \boldsymbol{r}|}\, \sigma_2 \times \boldsymbol{k}\, e^{ikr_2}\, U_{0k}. \tag{52}$$

As $\boldsymbol{k}$ is small we get the dominating term proportional to $\dfrac{\boldsymbol{k}}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}$. Further we get

$$\begin{aligned} U_2'(\boldsymbol{r}_1) = -\frac{1}{(4\pi)^2}\int\!\!\int dV'\, dV''\, \frac{\cos \boldsymbol{k}|\boldsymbol{r}_1 - \boldsymbol{r}'|}{|\boldsymbol{r}_1 - \boldsymbol{r}'|}\, \mathrm{div}' \Big[\boldsymbol{S}_3(\boldsymbol{r}') \times \\ \times\, \mathrm{grad}'\, \frac{\cos k|\boldsymbol{r}' - \boldsymbol{r}''|}{|\boldsymbol{r}' - \boldsymbol{r}''|}\, \mathrm{div}''\, (\boldsymbol{S}_3(\boldsymbol{r}'') \times \mathrm{grad}\, U_0(\boldsymbol{r}'')) \Big] \end{aligned} \tag{53}$$

where $U_2'(\boldsymbol{r}_1)$ is the first term in (30).

Terms where $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are both in S_1, do not contain $\boldsymbol{r}_2$ i. e. they may be omitted, as they do not contain the distance between the sources. Terms where $\boldsymbol{r}'$ is in S_2 and $\boldsymbol{r}''$ in S_1 or vice versa, may be neglected as they contain higher powers than the second of $\dfrac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$, for if we insert in these terms the values of $\boldsymbol{S}_3(\boldsymbol{r}')$ and $\boldsymbol{S}_3(\boldsymbol{r}'')$ respectively, being essentially δ-functions, we get through partial integration factors as $\operatorname*{grad}_{r'=r_2} \dfrac{1}{|\boldsymbol{r}' - \boldsymbol{r}_1|}$, multiplied by $\operatorname*{grad}_{r'=r_2} \dfrac{1}{|\boldsymbol{r}' - \boldsymbol{r}''|}$ where $\boldsymbol{r}''$ is in S_1, or $\operatorname*{grad''\cdot grad''}_{r''=r_2} \dfrac{1}{|\boldsymbol{r}' - \boldsymbol{r}''|}$ where $\boldsymbol{r}'$ is in S_1. Thus we may limit ourselves to the terms where both $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are in S_2. Putting $\boldsymbol{S}_3(\boldsymbol{r}')$ as a δ-function and carrying out the integration with regard to $\boldsymbol{r}'$ we have

$$\Delta U_2'(\boldsymbol{r}_1) = \frac{i}{(4\pi)^2} \int_{S_2} dV'' \operatorname*{grad}_{r'=r_2} \frac{\cos k|\boldsymbol{r}_1 - \boldsymbol{r}'|}{|\boldsymbol{r}_1 - \boldsymbol{r}'|} \cdot \Bigg\{\sigma_2 \times$$

$$\times \operatorname*{grad}_{r=r_2} \frac{\cos k|\boldsymbol{r} - \boldsymbol{r}''|}{|\boldsymbol{r} - \boldsymbol{r}''|}\Bigg\} \operatorname{div}(\boldsymbol{S}_3(\boldsymbol{r}'') \times \boldsymbol{k}\, e^{i k r''}\, U_{0k}). \tag{54}$$

From (50) we put $\boldsymbol{S}_3(\boldsymbol{r}'') = f(r)\,\sigma_2$, $r = |\boldsymbol{r}'' - \boldsymbol{r}_2|$.

Further we put

$$\frac{1}{(4\pi)^2} \operatorname*{grad}_{r'=r_2} \frac{\cos k|\boldsymbol{r}_1 - \boldsymbol{r}'|}{|\boldsymbol{r}_1 - \boldsymbol{r}'|} \times \sigma_2 = \boldsymbol{v},$$

$$\sigma_2 \times \boldsymbol{k}\, e^{i k r_2}\, U_{0k} = \boldsymbol{w}.$$

Neglecting the variation of the factors containing $\boldsymbol{k}$ inside the source we get

$$\begin{aligned} \Delta U_2'(\boldsymbol{r}_1) &= i \int dV \left(\boldsymbol{v} \cdot \operatorname{grad} \frac{1}{r}\right) \operatorname{div}(f(r)\,\boldsymbol{w}) = \\ &= i \int dV \left(\boldsymbol{v} \cdot \operatorname{grad} \frac{1}{r}\right) (\boldsymbol{w} \cdot \operatorname{grad} f(r)) = \\ &= i \int dV \frac{\boldsymbol{v} \cdot \hat{r}}{r^2}\, \boldsymbol{w} \cdot \hat{r}\, f'(r). \end{aligned} \tag{55}$$

Putting for example the polar axis parallel to $\boldsymbol{v}$ we get

$$\Delta U_2'(\boldsymbol{r}_1) = i \frac{4\pi}{3}\, \boldsymbol{w} \cdot \boldsymbol{v} \int_0^\infty f'(r)\, dr = i \frac{4\pi}{3}\, \boldsymbol{w} \cdot \boldsymbol{v}\, f(0).$$

If we assume that the source has constant density and the radius a i. e. if we put

$$f(r) = f(0) = \frac{3}{4\pi a^3} \tag{56}$$

we get

$$\varDelta U_2'(\boldsymbol{r}_1) = \frac{i}{(4\pi)^2 a^3} \left(\sigma_2 \times \operatorname*{grad}_{r=r_2} \frac{\cos k\,|\boldsymbol{r}_1 - \boldsymbol{r}|}{|\boldsymbol{r}_1 - \boldsymbol{r}|} \right) (\sigma_2 \times \boldsymbol{k}\, e^{i k r_2}). \tag{57}$$

As σ is of the order of magnitude of 1 the criterion for the validity of the perturbation calculation runs

$$\left| \frac{\gamma}{4\pi a^3} \right| \ll 1 \tag{58}$$

or from (49)

$$\left| \frac{2\, C_{s3}'}{\varkappa^3 a^3} \frac{e^2}{\hbar c} \right| \ll 1. \tag{59}$$

This is the same criterion as PAULI and HU obtained.

Inserting (28) and (45) in the second term of (30) we get

$$U_2''(\boldsymbol{r}_1) = -\frac{1}{4\pi} \int \frac{\cos k\,|\boldsymbol{r}_1 - \boldsymbol{r}'|}{|\boldsymbol{r}_1 - \boldsymbol{r}'|} \operatorname{div} \{ \boldsymbol{S}_3\, \boldsymbol{r}') \times (\boldsymbol{S}_3(\boldsymbol{r}') \times \operatorname{grad} U_0(\boldsymbol{r}'))\}\, d V'. \tag{60}$$

As above the integration is carried out only over S_2. One of the factors $\boldsymbol{S}_3(\boldsymbol{r}')$ can be put equal to $\boldsymbol{\sigma}_2\, \delta(\boldsymbol{r}' - \boldsymbol{r}_2)$ and the other $= \boldsymbol{\sigma}_2\, f(|\boldsymbol{r}' - \boldsymbol{r}_2|)$. Carrying out the integration of (60) we get

$$\varDelta U_2''(\boldsymbol{r}_1) = \frac{i}{4\pi} \operatorname*{grad}_{r=r_2} \frac{\cos k\,|\boldsymbol{r}_1 - \boldsymbol{r}|}{|\boldsymbol{r}_1 - \boldsymbol{r}|} \cdot f(0)\, (\boldsymbol{\sigma}_2 \times (\boldsymbol{\sigma}_2 \times \boldsymbol{k}))\, U_{0k}\, e^{i k r_2}. \tag{61}$$

Putting $f(0) = \dfrac{3}{4\pi a^3}$ and comparing (61) and (52) we get

$$\left| \frac{3}{4\pi a^3} \gamma \right| \ll 1$$

or from (51)

$$\left| 6 \frac{C_{s3}'}{\varkappa^3 a^3} \frac{e^2}{\hbar c} \right| \ll 1 \tag{62}$$

which is about the same criterion as (59).

B. *Vector mesons with spin-dependent interaction*

We assume also now that the interaction term is built up of antisymmetric tensors. The calculation is mainly the same as in the scalar case. The DIRAC equation runs (HJALMARS [49] 167, 192, 363)

$$\left[\left\{ \sum_{k=1}^{4} p_k \beta_k + i \mu c - i \gamma \mu c \sum_{\substack{k,l=1 \\ k \neq l}}^{4} \tilde{\psi}\, \gamma_k \gamma_l\, \psi\, \beta_k \beta_l\, \omega_5 \right\} U \right]_{mn} = 0 \tag{63}$$

where (HJALMARS [49] 161)

$$\omega_5 = \sum_{k-1}^{4} \beta_k^2 - 2$$

and (HJALMARS [49] 139. The index combinations 135 shall be used.)

$$(m\,n\,|\,\beta_k\,|\,m'\,n') = \tag{64}$$
$$= i\,(\delta_{mk}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'k}\,\delta_{nn'} - \delta_{nk}\,\delta_{m'5}\,\delta_{mn'} + \delta_{n5}\,\delta_{m'k}\,\delta_{mn'} -$$
$$- \delta_{mk}\,\delta_{n'5}\,\delta_{nm'} + \delta_{m5}\,\delta_{n'k}\,\delta_{nm'} + \delta_{nk}\,\delta_{n'5}\,\delta_{mm'} - \delta_{n5}\,\delta_{n'k}\,\delta_{mm'}).$$

We make the same assumptions as regards the source as in (38). Introducing (64) in (63) we get the following equations

$$\left.\begin{matrix} m \\ n \end{matrix}\right\} = 1, 2, 3, 4 \qquad p_m\,U_{5n} - p_n\,U_{5m} + \mu\,c\,U_{mn} = 0 \tag{65}$$

$$\sum_{k=1}^{3} p_k\,U_{k4} - \mu\,c\,U_{54} = 0 \tag{66}$$

and

$$n = 1, 2, 3 \qquad \sum_{k=1}^{4} p_k\,U_{kn} - \mu\,c\,U_{5n} + i\,\gamma\,\mu\,c\,(\boldsymbol{S}_3 \times \boldsymbol{U})_n = 0 \tag{67}$$

where

$$(\boldsymbol{U})_n = U_{5n}.$$

We multiply (66) and (67) by p_4 and p_n respectively and sum, remembering that $\sum_{n,k=1}^{4} p_n\,p_k\,U_{kn} = 0$ as U_{kn} is an antisymmetric tensor. We get then

$$\sum_{k=1}^{4} p_k\,U_{5k} - i\,\gamma \sum_{k=1}^{3} p_k\,(\boldsymbol{S}_3 \times \boldsymbol{U})_k = 0. \tag{68}$$

Inserting U_{mn} from (65) into (67) we get

$$n = 1, 2, 3 \qquad \sum_{k=1}^{4} p_k^2\,U_{5n} - p_n \sum_{k=1}^{4} p_k\,U_{5k} + \mu^2\,c^2\,U_{5n} - i\,\gamma\,\mu^2\,c^2\,(\boldsymbol{S}_3 \times \boldsymbol{U})_n = 0. \tag{69}$$

With the aid of (68) and introducing the definitions of p_k in (44) we get

$$\left(\Delta - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \varkappa^2\right)\boldsymbol{U} = i\,\gamma\,(\text{grad div} - \varkappa^2)\,(\boldsymbol{S}_3 \times \boldsymbol{U}). \tag{70}$$

Thus we have got the field equation in the form of (12) with

$$\Omega_1 = i\,(\text{grad div} - \varkappa^2)\,(\boldsymbol{S}_3 \times \ ) \tag{71}$$
$$\Omega_2 = 0.$$

As above we get the interaction energy (from 23, 27, 29, 71)

$$\Delta E_{2i} = \lim_{V=\infty} \frac{-1}{V} \sum_{\nu=1}^{3} \sum_{k',k''} \frac{\hbar c}{2} \frac{1}{\sqrt{k'^2 + \varkappa^2}} \cdot$$

$$\cdot \int_V \int_V \boldsymbol{U}^*_{0k'\nu} e^{-i\boldsymbol{k}'\boldsymbol{r}'} \left[(\mathrm{grad}' \, \mathrm{div}' - \varkappa^2) \left(\boldsymbol{S}_3(\boldsymbol{r}') \times \frac{e^{i\boldsymbol{k}''(\boldsymbol{r}'-\boldsymbol{r}'')}}{k'^2 - k''^2} \right) \cdot \right.$$

$$\left. \cdot (\mathrm{grad}'' \, \mathrm{div}'' - \varkappa^2) (\boldsymbol{S}_3(\boldsymbol{r}'') \times \boldsymbol{U}_{0k'\nu} e^{i\boldsymbol{k}'\boldsymbol{r}''}) \right] d V' \, d V'' \tag{72}$$

In order to evaluate these integrals we shall use the following vector relations. By partial integration and assuming that all surface integrals vanish we have[1]

$$\begin{aligned} &\int \boldsymbol{a} \cdot f(\boldsymbol{r}) \, \mathrm{grad} \, \mathrm{div} \, \boldsymbol{F}(\boldsymbol{r}) \, d V = \\ &= - \int \boldsymbol{a} \cdot \mathrm{grad} \, f(\boldsymbol{r}) \, \mathrm{div} \, \boldsymbol{F}(\boldsymbol{r}) \, d V = \\ &= \int (\boldsymbol{a} \, \mathrm{grad} \cdot \mathrm{grad} \, f(\boldsymbol{r})) \cdot \boldsymbol{F}(\boldsymbol{r}) \, d V \end{aligned} \tag{73}$$

and

$$\begin{aligned} &\int f(\boldsymbol{r}) \, \mathrm{grad} \, \mathrm{div} \, \boldsymbol{F}(\boldsymbol{r}) \, d V = \\ &= - \int \mathrm{grad} \, f(\boldsymbol{r}) \, \mathrm{div} \, \boldsymbol{F}(\boldsymbol{r}) \, d V = \\ &= \int \boldsymbol{F}(\boldsymbol{r}) \, \mathrm{grad} \cdot \mathrm{grad} \, f(\boldsymbol{r}) \, d V. \end{aligned} \tag{74}$$

As $\boldsymbol{S}_3(\boldsymbol{r}) = 0$ outside S_1 and S_2 these formulas may be used. We may put (cf. 36)

$$\boldsymbol{U}_{0k1} = \frac{\hat{x}}{\sqrt{V}} = \frac{\hat{x}_1}{\sqrt{V}}, \quad \boldsymbol{U}_{0k2} = \frac{\hat{y}}{\sqrt{V}} = \frac{\hat{x}_2}{\sqrt{V}}, \quad \boldsymbol{U}_{0k3} = \frac{\hat{z}}{\sqrt{V}} = \frac{\hat{x}_3}{\sqrt{V}} \tag{75}$$

assuming the same coordinate system as in Chapter I.

Using first (73) and then (74) we get

$$\Delta E_{2i} = \lim_{V=\infty} \frac{-1}{V^2} \sum_{k',k''} \sum_{\nu=1}^{3} \frac{\hbar c}{2} \frac{1}{\sqrt{k'^2 + \varkappa^2}} \frac{1}{k'^2 - k''^2} \cdot$$

$$\cdot \int_V \int_V dV' dV'' \, [\hat{x}_\nu \, (\mathrm{grad} \cdot \mathrm{grad} \; e^{-i\boldsymbol{k}'\boldsymbol{r}'} - \varkappa^2 e^{-i\boldsymbol{k}'\boldsymbol{r}'})] \cdot [\boldsymbol{S}_3(\boldsymbol{r}') \times$$

$$\times e^{i\boldsymbol{k}''\boldsymbol{r}'} \{ (\boldsymbol{S}_3(\boldsymbol{r}'') \times \hat{x}_\nu e^{i\boldsymbol{k}'\boldsymbol{r}''}) (\mathrm{grad} \cdot \mathrm{grad} \; e^{-i\boldsymbol{k}''\boldsymbol{r}''} - \varkappa^2 e^{-i\boldsymbol{k}''\boldsymbol{r}''}) \}]. \tag{76}$$

Analogous to (48) we get

$$\Delta E_{2i} = - \frac{\hbar c}{2} \frac{1}{(2\pi)^6} \int d\boldsymbol{k}' \, d\boldsymbol{k}'' \, d V' \, d V'' \frac{1}{\sqrt{k'^2 + \varkappa^2}} \frac{1}{k'^2 - k''^2} e^{i(\boldsymbol{k}' - \boldsymbol{k}'')(\boldsymbol{r}'' - \boldsymbol{r}')} \cdot$$

$$\cdot \{ \boldsymbol{S}_3(\boldsymbol{r}') \times [(\hat{x}_\nu \cdot \boldsymbol{k}') \, \boldsymbol{k}' + \varkappa^2 \hat{x}_\nu] \} \cdot$$

$$\cdot \{ [(\boldsymbol{S}_3(\boldsymbol{r}'') \times \hat{x}_\nu) \cdot \boldsymbol{k}''] \, \boldsymbol{k}'' + \varkappa^2 (\boldsymbol{S}_3(\boldsymbol{r}'') \times \hat{x}_\nu) \}. \tag{77}$$

---

[1] $\boldsymbol{a} \, \mathrm{grad} \cdot \boldsymbol{F}(\boldsymbol{r}) = a_x \frac{\partial \boldsymbol{F}}{\partial x} + a_y \frac{\partial \boldsymbol{F}}{\partial y} + a_z \frac{\partial \boldsymbol{F}}{\partial z}$.

If we put

$$\gamma = 8\pi\frac{e^2}{\hbar c}\frac{1}{\varkappa^3}C''_{v3}, \tag{78}$$

we get the same value for the interaction energy as in Chapter I $V_{3''3''}$. This can easily be seen. We may for example, having introduced the same polar coordinates as in Chapter I and exchanged the coordinates as in (48), carry out the integrations with respect to φ' and φ''. We get then the same expression as in Chapter I (51 $V_{3''3''}$), where we have neglected the terms containing $\boldsymbol{S}_2(\boldsymbol{r})$.

We shall now investigate the criterion in the case of vector mesons. We shall make the same assumptions as in the scalar case but we may now put $\boldsymbol{k} = 0$.

From (15), (28), (29), (30), and (71) we get

$$\boldsymbol{U}_1(\boldsymbol{r}_1) = -\left(\frac{i}{4\pi}\right)\int\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}'|}(\text{grad}'\,\text{div}' - \varkappa^2)(\boldsymbol{S}_3(\boldsymbol{r}')\times\boldsymbol{U}_0)\,dV' \tag{79}$$

and

$$\begin{aligned}\boldsymbol{U}_2(\boldsymbol{r}_1) = -\left(\frac{1}{4\pi}\right)^2\int\int\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}'|}\Big\{(\text{grad}'\,\text{div}' - \varkappa^2)\Big[\boldsymbol{S}_3(\boldsymbol{r}')\times\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\\ \cdot(\text{grad}''\,\text{div}'' - \varkappa^2)(\boldsymbol{S}_3(\boldsymbol{r}'')\times\boldsymbol{U}_0)\Big]\Big\}\,dV'\,dV''\end{aligned} \tag{80}$$

As in (52) and using (74) we get

$$\varDelta\,\boldsymbol{U}_1(\boldsymbol{r}_1) = -\frac{i}{4\pi}(\sigma_2\times\boldsymbol{U}_0)\left(\text{grad}\cdot\underset{r=r_2}{\text{grad}}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|} - \varkappa^2\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right). \tag{81}$$

In $\boldsymbol{U}_2(\boldsymbol{r}_1)$ we may as above omit terms where $\boldsymbol{r}'$ and $\boldsymbol{r}''$ are both in S_1 as these terms do not contain the distance between the sources. Further we may neglect those terms where $\boldsymbol{r}'$ is in S_2 and $\boldsymbol{r}''$ in S_1 for if we put $\boldsymbol{S}_3(\boldsymbol{r})$ as a δ-function and use (74) we see that these terms contain $\text{grad}\cdot\underset{r=r_2}{\text{grad}}\dfrac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|}$ multiplied by itself and thus contain higher powers than the third of $\dfrac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}$. Thus we may limit ourselves to discuss the terms where $\boldsymbol{r}'$ is in S_1 or in S_2 and $\boldsymbol{r}''$ is in S_2.

In the first case we may put $\boldsymbol{S}_3(\boldsymbol{r}'') = \sigma_2\,\delta(\boldsymbol{r}''-\boldsymbol{r}_2)$ and $\boldsymbol{S}_3(\boldsymbol{r}') = \sigma_1 f(r)$ where $r = |\boldsymbol{r}'-\boldsymbol{r}_1|$; in the second case we may put $\boldsymbol{S}_3(\boldsymbol{r}') = \sigma_2\,\delta(\boldsymbol{r}'-\boldsymbol{r}_2)$ and $\boldsymbol{S}_3(\boldsymbol{r}'') = \sigma_2 f(r)$ where $r = |\boldsymbol{r}''-\boldsymbol{r}_2|$.

Neglecting terms of higher power of $\dfrac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}$ than the third or $\varkappa^2$ times the first and using (74) we get

$$\begin{aligned}\varDelta\,\boldsymbol{U}_2(\boldsymbol{r}_1) = -\left(\frac{1}{4\pi}\right)^2\int\limits_{S_1} dV'\,\frac{1}{r}(\text{grad}'\,\text{div}' - \varkappa^2)\,f(r)\,\cdot\\ \cdot\left[\sigma_1\times\left\{(\sigma_2\times\boldsymbol{U}_0)\left(\text{grad}\cdot\underset{r=r_2}{\text{grad}}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|} - \varkappa^2\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)\right\}\right] -\end{aligned}$$

$$-\left(\frac{1}{4\pi}\right)^2 \int\limits_{S_2} d\,V'' \left(\text{grad}\cdot\underset{r=r_2}{\text{grad}}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|}-\varkappa^2\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)\cdot$$

$$\cdot\left[\sigma_2\times\frac{1}{r}\,\{(\text{grad}''\ \text{div}''-\varkappa^2)\,f(r)\,(\sigma_2\times\boldsymbol{U}_0)\}\right]\cdot \tag{82}$$

We have, however,

$$\text{grad div}\,(f(r)\,\boldsymbol{v})=\boldsymbol{v}\ \text{grad}\cdot\text{grad}\ f(r) \tag{83}$$

and

$$\int\frac{\boldsymbol{v}}{r}\,\text{grad}\cdot\text{grad}\ f(r)\,d\,V=$$

$$=\int\frac{\boldsymbol{v}}{r}\cdot\text{grad}\frac{f'(r)}{r}\,\boldsymbol{r}\,d\,V+\int\frac{\boldsymbol{v}}{r^2}f'(r)\ \text{grad}\cdot\boldsymbol{r}\,d\,V=$$

$$=\int\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^2}\,\frac{d}{d\,r}\left(\frac{f'(r)}{r}\right)\boldsymbol{r}\,d\,V+\boldsymbol{v}\int\frac{f'(r)}{r^2}\,d\,V=$$

(if we use polar coordinates with $\boldsymbol{v}$ as the polar axis)

$$=\frac{4\pi}{3}\,\boldsymbol{v}\int\limits_0^\infty\frac{d}{d\,r}\left(\frac{f'(r)}{r}\right)r^2\,d\,r+4\pi\,\boldsymbol{v}\int\limits_0^\infty f'(r)\,d\,r=$$

$$=\frac{4\pi}{3}\,\boldsymbol{v}\int\limits_0^\infty f'(r)\,r-\frac{8\pi}{3}\,\boldsymbol{v}\int\limits_0^\infty f(r)+4\pi\,\boldsymbol{v}\int\limits_0^\infty f(r)= \tag{84}$$

$$=-\frac{4\pi}{3}\,\boldsymbol{v}\,f(0)=$$

$\left(\text{if we assume } f(r) \text{ constant}=\frac{3}{4\pi a^3}\right)$

$$=-\frac{\boldsymbol{v}}{a^3}$$

and

$$\int\frac{1}{r}\,f(r)\,d\,V=4\pi\int\limits_0^\infty r\,f(r)\,d\,r=-\frac{3}{2\,a}\cdot \tag{85}$$

Thus

$$\Delta\,\boldsymbol{U}_2(\boldsymbol{r}_1)=\left(\frac{1}{4\pi}\right)^2\left(\frac{1}{a^3}+\frac{3\varkappa^2}{2\,a}\right)\cdot$$

$$\cdot\left[\sigma_1\times\left\{(\sigma_2\times\boldsymbol{U}_0)\left(\text{grad}\cdot\underset{r=r_2}{\text{grad}}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|}-\varkappa^2\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)\right\}+\right. \tag{86}$$

$$\left.+\ \{\sigma_2\times(\sigma_2\times\boldsymbol{U}_0)\}\left(\text{grad}\cdot\underset{r=r_2}{\text{grad}}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|}-\varkappa^2\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)\right]\cdot$$

Thus we get from (86) and (81)

$$\left|\frac{\gamma}{4\pi a^3}\right| \leqslant 1 \quad \text{and} \quad \left|\frac{3\varkappa^2\gamma}{4\pi\cdot 2a}\right| \leqslant 1$$

or from (78)

$$\left|\frac{2C''_{v3}}{\varkappa^3 a^3}\left(\frac{e^2}{\hbar c}\right)\right| \leqslant 1 \tag{87}$$

and

$$\left|\frac{3C''_{v3}}{\varkappa a}\left(\frac{e^2}{\hbar c}\right)\right| \leqslant 1. \tag{88}$$

If $(\varkappa a) < \sqrt{\tfrac{2}{3}}$ (87) is the stronger criterion, which is usually the case. (87) corresponds to the criterion in the scalar case with spin-dependent interaction and to the criterion found by PAULI und HU [45] with a spin-independent interaction in the vector case and (88) corresponds to the criterion that WENTZEL [42] and KLEIN found in the scalar case with scalar interaction. As we saw above (87) arises from the terms containing two differentiations with respect to space and (88) arises from the terms that contain $\varkappa^2$ as a factor, i. e. that do not contain any differentiations with respect to space.

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